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Algebraic Aspects of Tiling Semigroups

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To C., L., and P.

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Abstract

This thesis is devoted to the algebraic study of tiling semigroups, in the context of inverse semigroup theory.

Tiling semigroups were originally motivated by the work of Johannes Kellendonk [23], in connection with a problem in solid state physics, formulated in terms of almost-groupoids by the same author in [24], and established by Kellendonk and Mark V. Lawson in [26] and Lawson in [32]. Since then, quite a lot of research has been done on this subject, mainly regarding tilings of the real line. In this dissertation, we aimed at furthering the study of one-dimensional tiling semigroups and extending the theory concerning this class to a special type of n-dimensional tilings, named n-dimensional hypercubic tilings. Following the tradition initiated by Lawson in [33], we often conduct our investigations in the more general setting of an inverse semigroup associated with a factorial language.

The first three chapters are essentially introductory. In Chapter 1, we recall some selected background material; in Chapter 2, we present and investigate a construction, called generalized Bruck-Reilly extension, which will clarify the connection between hypercubic tiling semigroups and Bruck-Reilly extensions; in Chapter 3, we define all the concepts involved in the construction of the tiling semigroup, give a complete review on the research conducted on this subject, and introduce the notion of n-dimensional hypercubic tiling.

In Chapter 4, we introduce the notions of language of an n-dimensional hypercubic tiling and of n-dimensional factorial language with the purpose of generalizing to n-dimensional hypercubic tiling semigroups a convenient representation of one-dimensional tiling semigroups in terms of a language associated with the tiling introduced by Lawson in [32]. We also present another representation of an n-dimensional hypercubic tiling semigroup as a Rees factor semigroup of a subsemigroup of a generalized Bruck-Reilly extension. In Chapter 5, we develop a description of a tiling semigroup, both one-dimensional and hypercubic, as a P^*-semigroup; in Chapter 6, we compute a presentation for one-dimensional tilings semigroups and discuss some aspects of the presentability of n-dimensional hypercubic tiling semigroups; in Chapter 7, we provide a necessary and sufficient condition for hypercubic tilings to give rise to isomorphic tiling semigroups.

Keywords

(n-dimensional) factorial language, inverse semigroup associated with a factorial language, isomorphism, n-dimensional hypercubic tiling, one-dimensional tiling, P^*-semigroup, presentation of an inverse semigroup, tiling semigroup.

Resumo

Esta dissertação é dedicada a aspectos algébricos dos semigrupos de pavimentações, no âmbito da teoria dos semigrupos inversos.

Os semigrupos de pavimentações tiveram a sua origem num trabalho de Johannes Kellendonk relacionado com um problema da Física do Estado Sólido [23]. Neste trabalho, Kellendonk procedeu à aplicação de uma abordagem proposta por Jean Bellissard [4] cerca de uma década antes; a mais importante contribuição de Kellendonk foi o recurso a pavimentações como modelos discretos de sólidos. Por pavimentação de dimensão n entenda-se um conjunto infinito numerável de azujelos que cobrem $\mathbb{R}^n$ intersectando-se, quando muito, nas suas fronteiras. Na sua proposta, conhecida como classificação K-teórica de hiatos, Bellissard formulou uma teoria que veio contornar a grande dificuldade em estudar directamente a álgebra dos observáveis associada ao movimento de uma partícula num sólido, devida à complexidade dos cálculos envolvidos. Desta teoria faz parte o estudo de uma álgebra-C^*, que se pode identificar com a álgebra dos observáveis, a qual Kellendonk obtém a partir da pavimentação.

Posteriormente, Kellendonk adoptou uma abordagem diferente, ao construir a referida álgebra-C^* tendo como ponto de partida um quase-grupóide associado à pavimentação [24]. A introdução do semigrupo de uma pavimentação, tal como é até hoje considerado, deve-se a Lawson que, em [32] e num trabalho conjunto com Kellendonk [26], formula o quase-groupóide associado à pavimentação como um semigrupo inverso. Resumidamente, os elementos não nulos do semigrupo de uma pavimentação são subconjuntos finitos e conexos de azulejos da pavimentação com dois azulejos assinalados, identificados por translacção, e a operação definida entre eles assemelha-se à multiplicação das árvores de Munn. Não é por isso surpreendente que os semigrupos de pavimentações pertençam à importante classe dos semigrupos inversos E^*-unitários; de facto, são inclusivamente semigrupos inversos fortemente E^*-unitários. Desde que foram introduzidos, os semigrupos de pavimentações têm despertado o interesse e o trabalho de diversos autores, nomeadamente Dombi e Gilbert, Masuda e Morita, McAlister, e Zhu. Em muitos dos trabalhos destes autores é dada particular enfâse aos semigrupos de pavimentações de dimensão 1.

No nosso trabalho, procurámos aprofundar o estudo dos semigrupos de pavimentações

de dimensão 1 e generalizar a respectiva teoria a uma classe especial de pavimentações de dimensão n, ditas pavimentações hipercúbicas, que, para $n = 1$, coincidem com as pavimentações de dimensão 1. Relativamente ao semigrupo de uma pavimentação de dimensão 1 e ao semigrupo de uma pavimentação hipercúbica de dimensão n, os principais objectivos foram a obtenção de uma representação do semigrupo como um P^*-semigrupo, o estudo de aspectos relacionados com a apresentação destes semigrupos e a determinação das condições em que os mesmos são isomorfos. Em seguida, daremos uma breve descrição dos capítulos em que este texto está organizado.

Os primeiros três capítulos são essencialmente introdutórios. Do Capítulo 1 consta uma selecção de conceitos que, ainda que bem conhecidos, nem sempre são abordados, ou não o são de forma consensual, na bibliografia de referência da teoria dos semigrupos e dos semigrupos inversos, tal como é o caso dos P^*-semigrupos, das apresentações e das linguagens. Relativamente aos P^*-semigrupos, é descrito um processo de construção de uma representação de um semigrupo inverso fortemente E^*-unitário como um P^*-semigrupo a partir de um pré-homomorfismo idempotente-puro 0-restringido do semigrupo para um grupo com zero. Relativamente a apresentações, além de serem recordados os conceitos fundamentais e estabelecida alguma notação, são demonstrados resultados que se prendem com a discussão da finitude ou infinitude da apresentação de um semigrupo. No que respeita a linguagens, são definidas algumas classes de linguagens que irão surgir adiante relacionadas com o estudo de pavimentações de dimensão 1.

No Capítulo 2 é apresentada e estudada uma construção que generaliza a extensão de Bruck-Reilly, de um grupo por um endomorfismo, a qual será posteriormente usada para clarificar a relação entre as extensões de Bruck-Reilly e os semigrupos de pavimentações hipercúbicas. Foi de facto a semelhança entre as operações nestes definidas que motivou a formulação da nova extensão.

Quanto ao Capítulo 3, são dadas aqui todas as definições envolvidas na noção de pavimentação de dimensão n e de semigrupo de uma pavimentação. Fazemos ainda um resumo do trabalho desenvolvido por outros autores sobre este tema no âmbito da teoria dos semigrupos inversos. Por fim, é apresentada a classe de pavimentações hipercúbicas de dimensão n.

Devido às características particulares das pavimentações de dimensão 1, nomeadamente a de poderem ser identificadas com palavras bi-infinitas, o semigrupo $S(\mathcal{T})$ associado a uma pavimentação $\mathcal{T}$ de dimensão 1 é geralmente representado por meio de um semigrupo $S(L(\mathcal{T}))$ definido à custa da linguagem $L(\mathcal{T})$ constituída por todos os factores da palavra bi-infinita. A esta linguagem dá-se o nome de linguagem da pavimentação e facilmente se observa que é uma linguagem factorial, pois contém qualquer factor de qualquer palavra a ela pertencente. Além disso, mostra-se que, partindo de uma linguagem factorial arbitrária L, é possível considerar um semigrupo inverso $S(L)$, designado semigrupo inverso associado à linguagem factorial L, que generaliza o semigrupo de uma pavimentação 1, no sentido em que coincide com este quando a linguagem em causa é a linguagem da pavimentação. Esta representação de semigrupo de uma pavimentação de dimensão 1, devida a Lawson [32], é muito conveniente, na medida em que usa uma representação muito mais simples dos elementos do semigrupo, sem recurso expresso a classes de equivalência, e por isso usada em todas as investigações relacionadas com semigrupos de pavimentações de dimensão 1. No Capítulo 4, procedemos à generalização da representação do semigrupo de uma pavimentação hipercúbica de dimensão n através de uma linguagem factorial, para o que começamos por formular os conceitos de linguagem de uma pavimentação hipercúbica de dimensão n e de linguagem factorial de dimensão n. Em toda a nossa investigação, os resultados serão apresentados no caso mais geral dos semigrupos inversos associados a uma linguagem factorial sempre que tal for possível e relevante, ou feita a respectiva comparação com os semigrupos de pavimentações. Usando a representação do semigrupo de uma pavimentação hipercúbica como semigrupo associado a uma linguagem factorial, concretizamos, ainda neste capítulo, a relação entre semigrupos de pavimentações hipercúbicas e extensões de Bruck-Reilly, pela obtenção de uma representação dos primeiros como quociente de Rees de um subsemigrupo da extensão de Bruck-Reilly generalizada anteriormente desenvolvida.

No Capítulo 5 constrói-se uma representação do semigrupo de pavimentação hipercúbica de dimensão n, ou mais geralmente, do semigrupo inverso associado a uma linguagem factorial de dimensão n, como P^*-semigrupo, usando o processo descrito no Capítulo 1. Relativamente a este tópico, a semelhança entre o caso $n = 1$ e o caso $n \geqslant 2$ não podia ser maior. São ainda dados alguns exemplos que aplicam a representação obtida ao estudo de isomorfismos entre P^*-semigrupos e a sua relação com isomorfismos entre os P-semigrupos correspondentes.

No Capítulo 6, procedemos ao estudo da apresentação de um semigrupo de uma pavimentação hipercúbica. Ao contrário do tópico anterior, o estudo de apresentações põe em evidência alguns contrastes entre as classes de semigrupos de pavimentações consideradas. Para o semigrupo de uma pavimentação de dimensão 1, calculamos uma apresentação do semigrupo e caracterizamos, à custa da noção de palavra minimal proibida, os semigrupos de pavimentações de dimensão 1 que são finitamente apresentados. Em particular, mostramos que todo o semigrupo associado a uma pavimentação periódica (de dimensão 1) é finitamente apresentado e determinamos em que condições é que o mesmo sucede para pavimentações ultimamente periódicas. Já relativamente aos semigrupos de pavimentações hipercúbicas de dimensão $n \geqslant 2$, mostramos que nunca são finitamente apresentados, nem mesmo como semigrupos fortemente E^*-unitários que admitem um pré-homomorfismo idempotente-puro 0-restringido para um grupo abeliano com zero. Mostramos ainda que mesmo uma pavimentação periódica (hipercúbica de dimensão n) pode exigir um número infinito de relações para além daquelas que definem o semigrupo de uma pavimentação hipercúbica de dimensão n contendo todos os padrões sobre o mesmo alfabeto.

No Capítulo 7 determina-se uma condição necessária e suficiente para que semigrupos de pavimentações hipercúbicas de dimensão n sejam isomorfos e, também aqui, a noção

de linguagem da pavimentação revela-se muito apropriada. Em particular para $n = 1$, a caracterização obtida traduz-se na seguinte condição: ou as linguagens das pavimentações são a mesma a menos de uma bijecção entre os respectivos alfabetos ou, para além disso, as linguagens são o reverso uma da outra (ou seja, cada linguagem contém exactamente as palavras da outra escritas da direita para a esquerda).

Por fim, incluímos um Apêndice onde estudamos o grupo que desempenhou um importante papel no estudo dos isomorfismos entre semigrupos de pavimentações, chamado grupo completo das simetrias de um hipercubo (de dimensão n) (em inglês, *full symmetry group of the (n-dimensional) hypercube*). São apresentados resultados relativos à ordem e aos geradores do grupo, desenvolvida uma representação matricial para os seus elementos e demonstrada uma decomposição do grupo no produto semidirecto do grupo simétrico pelo grupo aditivo $\mathbb{Z}_2^n$. Ainda que alguns destes resultados sejam conhecidos, o seu tratamento raramente é desenvolvido nos textos de referência em teoria dos grupos.

Uma lista de questões motivadas pela investigação desenvolvida e presentemente em aberto encerra esta dissertação.

Palavras-chave

Apresentação de um semigrupo inverso, isomorfismo, linguagem factorial (de dimensão n), pavimentação de dimensão 1, pavimentação hipercúbica de dimensão n, P^*-semigrupo, semigrupo de uma pavimentação, semigrupo inverso associado a uma linguagem factorial,

Contents

List of Figures

Introduction

Tiling semigroups is now a 15 year old subject. Its origins go back to 1995 in a work by Johannes Kellendonk [23], where the author developed an approach to the study of the motion of a particle in a solid proposed by Jean Bellissard in 1982 [4].

Bellissard's proposal, known as "K-theoretical gap labelling", gave a method to study the gap labelling of the Schrödinger operators describing the motion of a particle in a solid. It was motivated by the extreme difficulty in obtaining any valuable information directly from the algebra of observables, due to the complexity of the calculations involved, and it consisted in the formulation of the gap labelling as a map from the gaps to a K_0-group of a C^*-algebra. This C^*-algebra could be interpreted as the algebra of observables, thus having a clear meaning in terms of the physical problem.

Kellendonk's pioneering contribution was the use of tilings as discrete models of solids. In a rather informal way, we shall for now think of a tiling of the Euclidean space $\mathbb{R}^n$ as a countable collection of subsets of $\mathbb{R}^n$ (the tiles of the tiling) whose union is $\mathbb{R}^n$ and which overlap, at most, at their boundaries. Most interestingly, Penrose tilings [49] — famous because of their mathematical properties, such as the lack of translational symmetry — waited ten years to be found as the ideal models for certain quasicrystals [35]. Kellendonk's application consisted in the construction of a non-commutative space and the corresponding C^*-algebra from a tiling.

Two other papers by Kellendonk would follow. In [24], Kellendonk takes a different approach to the same problem. This time, the definition of a multiplicative structure on the classes of portions of the tiling, consisting of a finite union of tiles under translation, yield an almost-groupoid from which a topological groupoid is built. The reason why only the local structure of the tiling (that is, how the tiling looks on finite patches) matters has to do with the local nature of the interactions in the solid. Subsequently, a group assigned to this topological groupoid is going to provide, in theory at least, some information on the solid modelled by the tiling. The almost-groupoid associated to the tiling is precisely the ancestor of the tiling semigroup, a concept that would only be formulated later by Mark V. Lawson [32] (see also Kellendonk and Lawson [26]).

In [25], Kellendonk furthers the study of tilings using the same approach as in [24], that is, using the almost-groupoid associated to the tiling. More precisely, Kellendonk investigates under which circumstances two tilings give rise to isomorphic topological groupoids.

Kellendonk's topological groupoid of an arbitrary almost-groupoid consists of the minimal elements among the equivalence classes of all decreasing sequences of elements from the almost-groupoid, for the equivalence relation that identifies sequences which have the same limit. Equipped with a suitable topology, the groupoid turns into a topological groupoid. This construction can also be found in Lawson's book [32], from a more semigroup-theoretic point a view and with a few proofs in more detail. Kellendonk's main result characterizes tilings whose almost-groupoids give rise to isomorphic topological groupoids in terms of the somewhat intricate notion of local derivability of tilings.

The current formulation of tiling semigroup is an adaptation of Kellendonk's almost-groupoid associated to a tiling and was, as mentioned above, introduced by Lawson in [32]. It is note worthly that, since every element in Kellendonk's almost-groupoid has a unique inverse and since Lawson's tiling semigroup is always an inverse semigroup, the difference between considering an almost-groupoid or an inverse semigroup is simply a matter of considering a partial operation or adjoining a zero. The main difference was that, in Kellendonk's almost-groupoid associated to a tiling, it is not required that the finite unions of tiles that underlie its elements constitute a connected subset of the space. From Lawson's [32] account of tiling semigroups (including Kellendonk and Lawson's joint work [26]), such elements are assumed to be connected unions. In [32], Lawson establishes the first known properties of tiling semigroups, of which the most important is, probably, that they are E^*-unitary.

Since Kellendonk's pioneering work on this subject, several authors have taken an interest, and from different points of view. An overview of the construction of C^*-algebras from tilings can be found in [27]. More generally, the study of groupoids associated with inverse semigroups and their operator algebras is extensively developed, and has in Alan Paterson's book [48] its fundamental reference. In fact, much of the work in this area applies either Paterson's universal groupoid of an inverse semigroup or a variation of it accordingly to the situation under investigation.

Kellendonk points out some differences between his and Paterson's groupoid, but does not go into making their relationship precise. In [34], Daniel Lenz considers two groupoids constructed from an inverse semigroup. The first consists of the equivalence classes of a certain equivalence relation defined on the set of down directed subsets of the semigroup, which, with the topology considered, turns out to be isomorphic to Paterson's groupoid. For this reason, it is named the universal groupoid of an inverse semigroup. The second, named minimal groupoid of an inverse semigroup, is the subgroupoid of the first consisting of its minimal elements, equipped with the induced topology. If, for an inverse semigroup S with zero, the minimal groupoid of S always agrees with Kellendonk's groupoid as a set, the topology of the minimal groupoid need not coincide with that of Kellendonk's. However, it is proved that the topologies do coincide when S satisfies the following condition: for all $x, y \in S$ such that there exists $z \in S$ (non-zero in case S has a zero) with $z \leqslant x, y$, then x and y have greatest lower bound in S. Since this condition is in fact a weaker version of the E^*-unitarity condition, Lenz thus arrives at an alternative description of Kellendonk's topological groupoid of a tiling. In addition, the relationship between Kellendonk's and Paterson's topological groupoids is, for semigroups satisfying the condition above, made clear.

Lenz's contribution no doubt consists in the most algebraic study of tilings with respect to the C^*-algebras relevant to the physical study intended. However, the gap between the tiling semigroup (or, equivalently, the almost-groupoid associated with the tiling) and its topological and physical counterparts remains too wide. And even in this dissertation, the focus relies exclusively on the algebraic study of tiling semigroups. Although this option may seem to leave out too much, we believe that tiling semigroups remain quite interesting mathematical objects — for us, they unquestionably constitute important examples of inverse semigroups. This opinion has found echo in several other authors, namely Dombi and Gilbert, Lawson, Masuda and Morita, McAlister, and Zhu, but we shall leave the overview on the literature on tiling semigroups from the algebraic point of view for after the definition of tiling and tiling semigroup in Chapter 3, where it can be best appreciated.

Thus, the aim of our investigations was to study algebraic aspects of tiling semigroups, within the context of inverse semigroup theory. We now outline the organization and main achievements presented in this thesis.

Three introductory chapters precede our investigations.

In Chapter 1, we present a rather selective set of background material, concerning the theory regarding the description of a strongly E^*-unitary inverse semigroup as a P^*-semigroup, presentations, and languages. All material that is given careful and uniform attention in the standard reference texts on semigroup and inverse semigroup theory [9, 10, 21, 32, 51] has been left out; brief reminders of some definitions and a few results will be given in the course of this work when they are called upon, for the convenience of the reader. The topics mentioned (which constitute, in that order, the three sections of this chapter), do not belong to that category; and although their contents are mostly known results, it was thought that they would benefit from a careful treatment beforehand. Other notions that are usually absent from the list of publications above but which, instead of being transversal to this work as it is the case with the three previously mentioned, are rather specific to a particular section, have been postponed until they are necessary.

In Chapter 2, a generalization of the well-known Bruck-Reilly extension of a group by an endomorphism is considered and its properties are studied. This construction was motivated by the resemblance between the operation in a Bruck-Reilly extension of a group and in a certain representation of the tiling semigroup associated with a special kind of n-dimensional tiling; in Section 4.3, that connection will be made precise. The reason for this topic being presented in an introductory stage of this thesis has to do with its fairly general nature.

Finally, Chapter 3 gives a careful definition of all concepts involved in the construction of the tiling semigroup and its origins, as well as a complete overview on the research conducted on this subject within the scope of inverse semigroup theory. Lastly, we define the notion of n-dimensional hypercubic tiling, which constitute the closest n-dimensional analogue to one-dimensional tilings (in fact, the class of one-dimensional hypercubic tilings can be identified with the class of one-dimensional tilings) and, thus, allows for a more detailed study than arbitrary tilings.

The achievements presented in this dissertation concern tiling semigroups associated with one-dimensional tilings and n-dimensional hypercubic tilings. Whenever possible and of interest, we also consider a generalization of these semigroups, namely that of an inverse semigroup associated with a factorial language, either proving our results in this setting or comparing the different behaviour of tiling semigroups and inverse semigroups associated with factorial languages. For the one-dimensional case, this generalization was introduced by Lawson. In [32], Lawson identifies a one-dimensional tiling $\mathcal{T}$ with a bi-infinite word and considers a certain language $L(\mathcal{T})$ associated to this word, called language of the tiling. The key property of these languages is that they are factorial, that is, contain every factor of a word in the language. He then shows how to represent the tiling semigroup $S(\mathcal{T})$ by means of $L(\mathcal{T})$, in the form of a semigroup denoted $S(L(\mathcal{T}))$. Moreover, he shows that, given an arbitrary factorial language L, we can define a semigroup $S(L)$, called inverse semigroup associated with the factorial language L, that coincides with the language respresentation $S(L(\mathcal{T}))$ of $S(\mathcal{T})$ when $L = L(\mathcal{T})$. Both in [32] and [33], Lawson conducts all his research regarding one-dimensional tiling semigroups in the context of an inverse semigroup associated with a factorial language.

For the case of n-dimensional hypercubic tilings, the analogue of this generalization is developed in Chapter 4. The reason for aspiring at a language representation of a tiling semigroup (associated with an n-dimensional hypercubic tiling) is that, in dimension 1, such a representation uses a much nicer representation of the elements of the semigroup, and consequently for conclusions that were not at hand in the case of arbitrary tilings and for which they could serve as an inspiration. In fact, even though one-dimensional tiling semigroups do not have any known physical applications, Kellendonk himself engages in its study in [23]. A primary aim regarding n-dimensional hypercubic tilings was, thus, to obtain an analogue of such a representation. Perhaps, the reason why arbitrary n-dimensional tiling semigroups are still too "far away" from being conveniently studied is that no such representation has yet been found for this case.

Therefore, Chapter 4 is dedicated to finding such a convenient description for n-dimensional hypercubic tiling semigroups, for which the notions of language of an n-dimensional hypercubic tiling and of n-dimensional factorial language are introduced in Section 4.1. In Section 4.2, we construct an inverse semigroup associated with an n-dimensional factorial language and prove that it yields a representation of the tiling semigroup associated with an n-dimensional hypercubic tiling, in case the factorial language is the language associated with the tiling. In addition, we show that it retrieves Lawson's language representation of one-dimensional tiling semigroup when $n = 1$. As already mentioned, motivated by the form of the multiplication in the language representation, we investigate in Section 4.3 the connection between hypercubic tiling semigroups and Bruck-Reilly extensions by means of the construction developed in Chapter 2.

Three main aspects of inverse semigroup theory were the object of our attention, concerning both one-dimensional and n-dimensional hypercubic tiling semigroups: (I) finding a representation of the semigroup as a P^*-semigroup; (II) the computation of a presentation for the semigroup, in the case of one-dimensional tilings, and a discussion of some aspects of the presentability of the semigroup, in the case of n-dimensional hypercubic tilings; (III) isomorphism conditions between tiling semigroups. Most significantly, the first topic relies entirely on the language derived from the tiling and the third, although it is not obtained in that way, turns out to have an interesting formulation in the language representation setting. With respect to the second topic, the investigations concerning one-dimensional tilings also rely exclusively on the tiling language, but the same approach proved not to be as interesting or justified for n-dimensional hypercubic tilings.

Chapter 5 deals with the first topic. The description of a tiling semigroup as a P^*-semigroup is obtained as the meeting point of several classical contributions by McAlister, Munn, and Reilly dating from the 1970's. In a nutshell, these are: McAlister's description of E-unitary inverse semigroups in the P-Theorem; Munn's constructive proof of this result; McAlister and Reilly's theory on E-unitary covers; McAlister's connection between E-unitary inverse semigroups and strongly E^*-unitary inverse semigroups. Let us outline the general idea. Firstly, recall that a strongly E^*-unitary inverse semigroup S comes equipped with a 0-restricted idempotent-pure pre-homomorphism from S into a group G with zero. This pre-homomorphism can be used to construct an E-unitary cover T for S, so that S is isomorphic to the Rees factor semigroup of T by an ideal I which is also fully determined in terms of G. On the other hand, T can be described as a P-semigroup $P(H, \mathcal{X}, \mathcal{Y})$ using Munn's proof of the P-Theorem, where H is the maximum group image of T, $\mathcal{X}$ is the direct product of H by the semilattice of idempotents E_T of T under an equivalence relation and $\mathcal{Y}$ is isomorphic to E_T; even better, in view of the connection between S and T, the description of T can be modified to be given in terms of G and of the semilattice of idempotents of S. Since the isomorphism from T onto $P(G, \mathcal{X}, \mathcal{Y})$ is explicitly known, so is the image of the ideal I under it and therefore we obtain an isomorphism from S onto a P^*-semigroup $P^*(G, \mathcal{X}^*, \mathcal{Y}^*)$. The general construction, described in Section 1.1, is applied to one-dimensional tiling semigroups in Section 5.1 and a direct proof of its generalization to n-dimensional hypercubic tilings constitutes Section 5.2. As an application of this representation in the one-dimensional case, Section 5.3 provides a few examples that illustrate some aspects of the theory on strongly E^*-unitary inverse semigroups, namely in what concerns isomorphisms between P^*-semigroups and their relationship with isomorphisms between the corresponding P-semigroups.

The second topic, concerning presentations, is developed in Chapter 6 and evidences the simpler nature of the structure of one-dimensional tiling semigroups when compared with n-dimensional hypercubic tiling semigroups, with $n \geqslant 2$. Here, we were able to produce a presentation for the first, but not for the second. Instead, investigations were made into the finite or the infinite presentability of n-dimensional hypercubic tiling semigroups, to conclude that, unlike one-dimensional tiling semigroups, these are always infinitely presented — not only as inverse semigroups, but also as strongly E^*-unitary inverse semigroups that admit a 0-restricted idempotent-pure pre-homomorphism into an abelian group with zero. Roughly speaking, this means that, for $n \geqslant 2$, every n-dimensional hypercubic tiling semigroup is infinitely presented, even if we discharge all relations that ensure that the elements of the form $uvu^{-1}v^{-1}$ are idempotents. Our strategy is, given an n-dimensional hypercubic tiling semigroup $S(\mathcal{T})$ with $n \geqslant 2$, to construct a finitely generated strongly E^*-unitary inverse semigroup $W_{\mathcal{T}}$ that admits a 0-restricted idempotent-pure pre-homomorphism into an abelian group with zero and an infinitely generated congruence ρ on $W_{\mathcal{T}}$ such that $S(\mathcal{T}) \simeq W_{\mathcal{T}}/\rho$. In view of a result proved in Chapter 1, this yields the desired conclusion. With respect to one-dimensional tiling semigroups, the presentation obtained throws an interesting light over the structure of these semigroups, by making explicit how its operation functions on two different levels, which we could regard as mechanical versus semantic. Contrary to n-dimensional hypercubic tiling semigroups with $n \geqslant 2$, one-dimensional tiling semigroups are finite or infinitely presented according to a property of the language of the tiling which concerns the shortest possible words, over the alphabet, that do not belong to the language. These words, so-called minimal forbidden words of the language, have proven useful before, namely in the context of Symbolic Dynamics [2]. In addition, it is shown using this notion that, in particular, one-dimensional periodic tilings always give rise to finitely presented semigroups (whereas n-dimensional periodic hypercubic tilings with $n \geqslant 2$ do not) and a characterization for those one-dimensional two-way ultimately periodic tilings semigroups which are finitely presented is also proven. The investigations concerning one-dimensional tilings constitute Section 6.1 and those concerning n-dimensional hypercubic tilings with $n \geqslant 2$ can be found in Section 6.2.

The third topic, in seeking the answer to the question of when two tiling semigroups are isomorphic, aims at singling out the main features of a tiling that a tiling semigroup is able to capture. And again, we prove in Chapter 7, Section 7.2, that the answer is in the language of the tiling: it is shown that, under mild conditions, two n-dimensional hypercubic tiling semigroups are isomorphic if and only if their languages are essentially the same, subject to a bijection between the alphabets and the action of a symmetry of the Eucledian space $\mathbb{R}^n$ of a special kind. A brief survey of the group consisting of the relevant symmetries just mentioned, known as the full symmetry group of the n-dimensional hypercube, is given in Section 7.1. In Section 7.3, we show that, when applied to tilings of the real line, this means that either one of the languages is the image of the other via the alphabet bijection or that, additionally, a reversal of all words also takes place.

Lastly, an Appendix has been added to this work to deal with the full symmetry group G_n of the n-dimensional hypercube, where we prove the results necessary to our application and some other interesting ones. We present results concerning its order and set of generators, a matrix representation and a decomposition as a semidirect product of the symmetric group by the additive group $\mathbb{Z}_2^n$. Although the results on the order and the semidirect product decomposition can be found in the literature [18], their proofs are not, to our understanding, sufficiently detailed, as it happens with most of the properties regarding G_n mentioned in the literature.

A list of some open questions closes this dissertation.

Chapter 1

Selected background

This chapter consists of a selected review of concepts and results necessary to our work. The criterion of our choice is the following: we have only included items that are either usually absent from the reference textbooks on semigroup or on inverse semigroup theories [9, 10, 21, 32, 51], or that are given different definitions in the literature, or which were thought to particularly benefit from a detailed treatment. Thus, all definitions and facts mentioned in the remaining chapters but not presented here can be found, and will often be referred to, in the books listed above. Occasionally, for the convenience of the reader we will briefly recall a specific definition or result when introducing a subject. In this chapter, we will recall some definitions, notation and results on P^*-semigroups, presentations, and languages.

1.1 On P*-semigroups

The class of E-unitary inverse semigroups and its analogue for semigroups with zero, the class of E^*-unitary inverse semigroups, are classical and very important in the context of inverse semigroup theory. For these classes, powerful structure theorems are known and many important special semigroup examples fit into them. Our interest in these classes lies in the fact that tiling semigroups are examples of E^*-unitary inverse semigroups; in fact, every tiling semigroup belongs to a notable subclass of E^*-unitary inverse semigroups, namely the class of strongly E^*-unitary inverse semigroups. In this section, after recalling the notions of E-unitary, E^*-unitary, and strongly E^*-unitary inverse semigroups, we state the structure theorems that describe them, prove in detail the construction of a P^*-semigroup that is isomorphic to a given strongly E^*-inverse semigroup (Construction 1.1.4), and investigate isomorphisms between E^*-unitary inverse semigroups (Theorem 1.1.6).

Given a semigroup S, we denote by E_S its set of idempotent elements. Recall that E_S is a semilattice when S is inverse. Also recall that, on an inverse semigroup S, the *natural partial order on S* is defined by, for all $s, t \in S$,

$$s \leqslant t \Leftrightarrow s = et \text{ for some } e \in E_S\,.$$

An inverse semigroup is *E-unitary* if, for each $s \in S$ and each $e \in E_S$, the condition $es \in E_S$ implies that $s \in E_S$. The corresponding notion for semigroups with zero is that of an *E^*-unitary* inverse semigroup, in which $0 \neq es \in E_S$ implies that $s \in E_S \setminus \{0\}$, for all $s \in S$ and $e \in E_S \setminus \{0\}$. An inverse semigroup with zero S is *strongly E^*-unitary* if it admits a *0-restricted idempotent-pure pre-homomorphism into a group with zero*, that is, if there exists a map $\lambda\colon S \to G^0$ into a group with a zero adjoined satisfying the following conditions:

(i) $s\lambda = 0$ if and only if $s = 0$ (λ is *0-restricted*),

(ii) $s\lambda = 1$ if and only if s is idempotent (λ is *idempotent-pure*),

(iii) for all $s, t \in S$ such that $st \neq 0$, we have $(st)\lambda = s\lambda t\lambda$ (λ is a *pre-homomorphism*).

Every strongly E^*-unitary inverse semigroup is E^*-unitary, but the converse is not true [6], and it was proved by Steinberg [59] that the question of knowing when an E^*-unitary inverse semigroup is strongly E^*-unitary is undecidable.

The structure of E-unitary inverse semigroups is completely determined by the so-called P-Theorem, due to McAlister. A few notions are necessary in order to state this result.

A partially ordered set $(\mathcal{X}, \leqslant)$ is *down directed* if, for all $x, y \in \mathcal{X}$ there exists $z \in \mathcal{X}$ such that $z \leqslant x$ and $z \leqslant y$. A subset $\mathcal{Y}$ of a partially ordered set $(\mathcal{X}, \leqslant)$ is an *order ideal of $\mathcal{X}$* if, whenever $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ are such that $x \leqslant y$, then $x \in \mathcal{Y}$. A partially ordered set $\mathcal{Y}$ is an *inf-semilattice* if, for all $x, y \in \mathcal{Y}$, the greatest lower bound $x \wedge y$ exists in $\mathcal{Y}$.

We say that a *group G acts (on the left) on a partially ordered set $\mathcal{X}$ by order automorphisms* if there exists a homomorphism of G into the group of order automorphisms of $\mathcal{X}$, that is, a map $\alpha\colon G \to \mathrm{Aut}\mathcal{X}$ such that $x(g\alpha) \leqslant y(g\alpha)$ whenever $x \leqslant y$, for all $g \in G$ and $x, y \in \mathcal{X}$.

Theorem 1.1.1 ([38], Theorem 2.6)**.** *Let $\mathcal{X}$ be a down directed partially ordered set, $\mathcal{Y}$ an inf-semilattice which is an order ideal of $\mathcal{X}$, and G a group which acts on $\mathcal{X}$ on the left by order automorphisms such that $G \cdot \mathcal{Y} = \mathcal{X}$. Let*

$$P(G, \mathcal{X}, \mathcal{Y}) = \big\{(x, g) \in \mathcal{Y} \times G\colon g^{-1} \cdot x \in \mathcal{Y}\big\}$$

and consider the following operation on $P(G, \mathcal{X}, \mathcal{Y})$: for all $(x, g), (y, h) \in P(G, \mathcal{X}, \mathcal{Y})$, set $(x, g)(y, h) = (x \wedge (g \cdot y), gh)$. Then $P(G, \mathcal{X}, \mathcal{Y})$ is an E-unitary inverse semigroup. Conversely, every E-unitary inverse semigroup is isomorphic to one of this form for unique G, $\mathcal{X}$ and $\mathcal{Y}$ up to isomorphism and equivalent group actions.

A semigroup $P(G, \mathcal{X}, \mathcal{Y})$ is called a *P-semigroup* and a triple $(G, \mathcal{X}, \mathcal{Y})$ with G, $\mathcal{X}$ and $\mathcal{Y}$ as in the statement of the P-Theorem became known as a *McAlister triple*.

The last assertion of the P-Theorem is a consequence of the following result:

Theorem 1.1.2 ([38], Theorem 1.3)**.** *The P-semigroups $P(G, \mathcal{X}, \mathcal{Y})$ and $P(H, \mathcal{U}, \mathcal{V})$ are isomorphic if and only if there exist a group isomorphism $\alpha\colon G \to H$ and an order* *isomorphism* $\Phi\colon \mathcal{X} \to \mathcal{U}$ *such that* $\mathcal{Y}\Phi = \mathcal{V}$ *and, for all* $g \in G$*, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{g\cdot} & \mathcal{X} \\ \Phi\big\downarrow & \circlearrowleft & \big\downarrow\Phi \\ \mathcal{U} & \xrightarrow[(g\alpha)\cdot]{} & \mathcal{U} \end{array}$$

that is, $(g \cdot x)\Phi = (g\alpha) \cdot x\Phi$*, for all* $x \in \mathcal{X}$*. Moreover, the isomorphism* ψ *from* $P(G,\mathcal{X},\mathcal{Y})$ *onto* $P(H,\mathcal{U},\mathcal{V})$ *is defined by* $(e,g)\psi = (e\Phi, g\alpha)$*.*

For G and H groups acting on partially ordered sets $\mathcal{X}$ and $\mathcal{U}$, respectively, $\alpha\colon G \to H$ a group isomorphism, and $\Phi\colon \mathcal{X} \to \mathcal{U}$ an order isomorphism, we say that (α, Φ) is a *compatible pair* if $(g \cdot x)\Phi = (g\alpha) \cdot x\Phi$, for all $g \in G$ and $x \in \mathcal{X}$.

Among several existing proofs of the P-Theorem, the one presented by Munn in [47] has the practical advantage of being constructive. Since we will make use of it, we recall it here. Let T be an E-unitary inverse semigroup. Take:

- $G = T/\sigma$, where

  $$\sigma = \{(s,t) \in T \times T\colon se = te \text{ for some } e \in E_T\}$$

  is the *minimum group congruence* on T;

- the equivalence relation $\equiv$ defined on $E_T \times G$ by

  $$(e,g) \equiv (f,h) \Leftrightarrow \exists\, t \in T \text{ such that } e = tt^{-1}, f = t^{-1}t, \text{ and } g^{-1}h = t\sigma^\natural\,,$$

  where $\sigma^\natural\colon T \to G = T/\sigma$ is the canonical epimorphism associated to σ;

- the partial order on $\mathcal{X} = (E_T \times G)/_\equiv$ defined by

  $$[e,g] \leqslant [f,h] \;\Leftrightarrow\; \exists\, a,b \in E_T, k \in G \text{ such that } \begin{cases} a \leqslant b \\ (a,k) \equiv (e,g) \\ (b,k) \equiv (f,h)\,; \end{cases}$$

- $\mathcal{Y} = \{[e,1]\colon e \in E_T\}$;

- the action of G on $\mathcal{X}$ given by, for all $h \in G$ and $[e,g] \in \mathcal{X}$,

  $$h \cdot [e,g] = [e,hg]\,.$$

Munn proved that $(G,\mathcal{X},\mathcal{Y})$ is a McAlister triple and that $T \simeq P(G,\mathcal{X},\mathcal{Y})$, via the isomorphism $\pi\colon T \to P(G,\mathcal{X},\mathcal{Y})$ defined by

$$t\pi = ([tt^{-1},1], t\sigma^\natural)\,,$$

for all $t \in T$, thus arriving at a description for T as a P-semigroup in terms of the semilattice E_T and the congruence σ.

Note that the equivalence relation $\equiv$ on $E_T \times G$ could be equivalently defined through the quasi-order $\preceq$ on this set given by

$$(e,g) \preceq (f,h) \Leftrightarrow \exists\, t \in T \text{ such that } e = tt^{-1}, t^{-1}t \leqslant f, \text{ and } g^{-1}h = t\sigma^{\natural}\,, \tag{1.1}$$

since

$$(e,g) \equiv (f,h) \Leftrightarrow (e,g) \preceq (f,h) \text{ and } (f,h) \preceq (e,g)\,.$$

Moreover,

$$[e,g] \leqslant [f,h] \Leftrightarrow (e,g) \preceq (f,h)$$

is the partial order on $\mathcal{X}$.

Much of the structure of strongly E^*-unitary inverse semigroups can be read off from that of E-unitary inverse semigroups, because not only is every Rees factor semigroup of an E-unitary inverse semigroup a strongly E^*-unitary inverse semigroup [6], but also because the converse holds as well: McAlister [42] and, independently, Margolis and Steinberg [59], have shown that strongly E^*-unitary inverse semigroups are Rees factor semigroups of E-unitary inverse semigroups. For instance, we have the following analogue of the P-Theorem for strongly E^*-unitary inverse semigroups:

Theorem 1.1.3 ([42], Theorem 3.1)**.** *Let $\mathcal{X}$ be a partially ordered set with minimum element 0, $\mathcal{Y}$ an inf-semilattice which is an order ideal of $\mathcal{X}$, and G a group which acts on $\mathcal{X}$ on the left by order automorphisms such that $G \cdot \mathcal{Y} = \mathcal{X}$. Let $\mathcal{X}^* = \mathcal{X} \setminus \{0\}$ and $\mathcal{Y}^* = \mathcal{Y} \setminus \{0\}$. Then the set*

$$P^*(G,\mathcal{X}^*,\mathcal{Y}^*) = \big\{(x,g) \in \mathcal{Y}^* \times G \colon g^{-1} \cdot x \in \mathcal{Y}^*\big\} \cup \{0\}$$

equipped with the operation defined by

$$(x,g)(y,h) = \begin{cases} (x \wedge (g \cdot y), gh), & \text{if } x \wedge (g \cdot y) \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

for all $(x,g),(y,h) \in P(G,\mathcal{X},\mathcal{Y})$, and all other products equal to 0, is a strongly E^*-unitary inverse semigroup. Conversely, every strongly E^*-unitary inverse semigroup is isomorphic to a semigroup $P^*(G,\mathcal{X}^*,\mathcal{Y}^*)$, for some (not necessarily unique) G, $\mathcal{X}$ and $\mathcal{Y}$.

Given G, $\mathcal{X}$ and $\mathcal{Y}$ as in the statement of the previous theorem, we call $(G,\mathcal{X},\mathcal{Y})$ a *McAlister* 0*-triple*, and $P^*(G,\mathcal{X}^*,\mathcal{Y}^*)$ a P^**-semigroup*.

Next, we outline McAlister's construction of a representation of a strongly E^*-unitary inverse semigroup as a P^*-semigroup [42]; because the fact that strongly E^*-unitary inverse semigroups are Rees factor semigroups of E-unitary inverse semigroups is an important step in this construction, we first outline the proof of this statement.

A few definitions and results are needed.

For clarity, for the remainder of this section we will denote the minimum group congruence on a semigroup T by σ_T. Let S be an inverse semigroup with zero, T an inverse semigroup, and G a group. We say that T is an *E-unitary cover for S over G* if T is E-unitary, G is isomorphic to the maximum group image T/σ_T of T, and there exists an *idempotent-separating epimorphism* $\eta\colon T \to S$, that is, an onto homomorphism η from T onto S such that, for all $e, f \in E_T$, if $e\eta = f\eta$, then $e = f$.

Let G be a group. Schein [56] showed that the set

$$\mathcal{K}(G) = \bigcup \{Hg\colon H \text{ is a subgroup of } G \text{ and } g \in G\} \ ,$$

equipped with the operation $Hg * Kh = (H \vee gKg^{-1})gh$, for all $Hg, Kh \in \mathcal{K}(G)$, is an inverse semigroup with zero G.

In [39, Theorem 3.9] McAlister and Reilly showed that given an inverse semigroup S, a group G and an idempotent-pure pre-homomorphism $\phi\colon S \to \mathcal{K}(G)$ such that $G = \bigcup\{x\phi\colon x \in S\}$, then the subsemigroup $T = \{(x, g) \in S \times G\colon g \in x\phi\}$ of the direct product $S \times G$ is an E-unitary cover for S over G and that, conversely, every E-unitary cover for S over G is isomorphic to a semigroup of this form.

McAlister's strategy for showing that a strongly E^*-unitary inverse semigroup is isomorphic to a Rees factor semigroup of an E-unitary inverse semigroup relies on the following idea: for a strongly E^*-unitary inverse semigroup S, with, say, $\lambda\colon S \to G^0$ a 0-restricted idempotent-pure pre-homomorphism, an E-unitary cover for S over G can be obtained from λ. In fact, if $\lambda\colon S \to G^0$ is such a mapping, then $\bar{\lambda}\colon S \to \mathcal{K}(G)$, defined by $0\bar{\lambda} = G$ and $x\bar{\lambda} = \{x\lambda\}$, for $x \in S \setminus \{0\}$, is an idempotent-pure pre-homomorphism for which $G = \bigcup \big\{x\bar{\lambda}\colon x \in S\big\}$. Thus, by McAlister and Reilly's result mentioned in the previous paragraph

$$\begin{aligned} T &= \big\{(x, g) \in S \times G\colon g \in x\bar{\lambda}\big\} \\ &= \{(x, x\lambda)\colon x \in S \setminus \{0\}\} \cup \{(0, g)\colon g \in G\} \end{aligned}$$

is an E-unitary cover for S over G, with $\eta\colon T \to S$, defined by $(x, g)\eta = x$, a surjective idempotent-separating homomorphism. Since

$$\begin{aligned} \ker \eta &= \{((x, g), (y, h)) \in T \times T\colon (x, g)\eta = (y, h)\eta\} \\ &= \{((x, g), (y, h)) \in T \times T\colon x = y\} \\ &= \{((x, g), (x, h))\colon x \in S \text{ and } g, h \in G\} \\ &= \{((x, x\lambda), (x, x\lambda))\colon x \in S \setminus \{0\}\} \cup \{((0, g), (0, h))\colon g, h \in G\} \\ &= \rho_I \end{aligned}$$

is the Rees congruence associated with the ideal $I = \{(0, g)\colon g \in G\}$ of T, we conclude that $S \simeq T/\rho_I = T/I$, as $S \simeq T/\ker \eta$. Hence, S is a Rees factor semigroup of an E-unitary inverse semigroup.

Construction 1.1.4. We now describe a way of constructing a representation of a given strongly E^*-unitary inverse semigroup as a P^*-semigroup. Let S be a strongly E^*-unitary inverse semigroup, with $\lambda\colon S \to G^0$ a 0-restricted idempotent-pure pre-homomorphism, and $T = \{(x, x\lambda)\colon x \in S \setminus \{0\}\} \cup \{(0,g)\colon g \in G\}$ the E-unitary cover for S over G considered by McAlister.

The first step is to use Munn's constructive proof of the P-Theorem as described before to find a description for T as a P-semigroup, which is given in terms of E_T and $G \simeq T/\sigma_T$. The second step is to translate it in terms of E_S and λ. Now,

$$(x,g)(x,g) = (x,g) \Leftrightarrow (x^2, g^2) = (x,g) \Leftrightarrow x^2 = x \text{ and } g^2 = g \Leftrightarrow x \in E_S \text{ and } g = 1\,,$$

so that $E_T = \{(e,1)\colon e \in E_S\} \simeq E_S$. Also, $(x,g)\sigma_T(y,h)$ if and only if $g = h$, since $T/\sigma_T \simeq G$ and S has a zero:

$$\begin{aligned}(x,g)\sigma_T(y,h) &\Leftrightarrow \exists\,(e,1) \in E_T \text{ such that } (x,g)(e,1) = (y,h)(e,1)\\ &\Leftrightarrow \exists\,(e,1) \in E_T \text{ such that } (xe,g) = (ye,h)\\ &\Leftrightarrow \exists\, e \in E_S \text{ such that } \ xe = ye \text{ and } g = h\\ &\Leftrightarrow g = h\,.\end{aligned}$$

But then, from rule (1.1),

$$\begin{aligned}((e,1),g) \preceq ((f,1),h) \Leftrightarrow\ & \exists\,(x,k) \in T \text{ such that } (e,1) = (x,k)(x,k)^{-1}, (f,1) \leqslant (x,k)^{-1}(x,k)\\ & \text{and } g^{-1}h = (x,k)\sigma_T^\natural\\ \Leftrightarrow\ & \exists\,(x,k) \in T \text{ such that } (e,1) = (xx^{-1},1), (f,1) \leqslant (x^{-1}x,1)\\ & \text{and } g^{-1}h = k\\ \Leftrightarrow\ & \exists\, x \in S \text{ such that } e = xx^{-1}, f \leqslant x^{-1}x \text{ and } g^{-1}h \in x\bar{\lambda}\,,\end{aligned}$$

and so

$$((e,1),g) \equiv ((f,1),h) \Leftrightarrow \exists\, x \in S,\ e = xx^{-1}, f = x^{-1}x \text{ and } g^{-1}h \in x\bar{\lambda}\,.$$

Notice that $((0,1),g) \equiv ((f,1),h)$ if and only if $f = 0$.

Therefore, we get the following description of T as a P-semigroup: $T \simeq P(G, \mathcal{X}, \mathcal{Y})$, with

- $\mathcal{X} = (E_S \times G)/_{\equiv}$, that is,

$$\mathcal{X} = \{[e,g]\colon g \in G \text{ and } e \neq 0\} \cup \{[0,1]\}$$

where, for all $(e,g), (f,h) \in \mathcal{X}$ with $e, f \neq 0$, the equivalence relation

$$(e,g) \equiv (f,h) \Leftrightarrow \exists\, x \in S \text{ such that } e = xx^{-1}, x^{-1}x = f \text{ and } g^{-1}h \in \{x\lambda\}$$

is obtained in the usual way from the quasi-order on $E_S \times G$ defined by

$$(e,g) \preceq (f,h) \Leftrightarrow \exists\, x \in S \text{ such that } e = xx^{-1}, x^{-1}x \leqslant f \text{ and } g^{-1}h \in \{x\lambda\}$$

and the partial order is given by

$$[0,1] \leqslant [e,g] \text{ and } [e,g] \leqslant [f,h] \Leftrightarrow (e,g) \preceq (f,h)\,,$$

for all $e,f \in E_S \setminus \{0\}$ and $g,h \in G$;

- $\mathcal{Y} = \{[e,1] : e \in E_S\}$;
- the action of G on $\mathcal{X}$ is given by

$$h \cdot [e,g] = [e,hg] \;;$$

- $P(G,\mathcal{X},\mathcal{Y}) = \big\{([e,1],g) \in \mathcal{Y} \times G \colon g^{-1} \cdot [e,1] \in \mathcal{Y}\big\}$;
- $\pi\colon T \to P(G,\mathcal{X},\mathcal{Y})$ defined by $(x,g)\pi = \big(\big[xx^{-1},1\big]\,,g\big)$ is the desired isomorphism.

As in the statement of Theorem 1.1.3, write $\mathcal{X}^*$ for $\mathcal{X} \setminus \{[0,1]\}$ and $\mathcal{Y}^*$ for $\mathcal{Y} \setminus \{[0,1]\}$. Hence, since

$$J = I\pi = \{(0,g) \colon g \in G\}\,\pi = \{([0,1],g) \colon g \in G\}\;,$$

then

$$P(G,\mathcal{X},\mathcal{Y}) \setminus J = \big\{([e,1],g) \in \mathcal{Y}^* \times G \colon g^{-1} \cdot [e,1] \in \mathcal{Y}^*\big\} = P^*(G,\mathcal{X}^*,\mathcal{Y}^*) \setminus \{0\}\,,$$

and we conclude that

$$S \simeq T/I \simeq P(G,\mathcal{X},\mathcal{Y})/J \simeq P^*(G,\mathcal{X}^*,\mathcal{Y}^*)$$

is a description of S as a P^*-semigroup, where a non-zero element $x \in S$ gets mapped to $([xx^{-1},1],x\lambda)$ and 0 gets mapped to the minimum element $\{([0,1],g) \colon g \in G\}$ of $P^*(G,\mathcal{X}^*,\mathcal{Y}^*)$.

The nice behaviour of E-unitary inverse semigroups concerning isomorphisms does not have an analogue for strongly E^*-unitary inverse semigroups. In Chapter 5, we will come across examples of this fact. It is however true that given any isomorphism φ between strongly E^*-unitary inverse semigroups S and T there are representations of S and T as P^*-semigroups for which φ arises from an isomorphism of the corresponding P-semigroups. Recall from Theorem 1.1.2 that the P-semigroups $P(G,\mathcal{X},\mathcal{Y})$ and $P(H,\mathcal{U},\mathcal{V})$ are isomorphic if and only if there exist a group isomorphism $\alpha\colon G \to H$ and an order isomorphism $\Phi\colon \mathcal{X} \to \mathcal{U}$ such that $\mathcal{Y}\Phi = \mathcal{V}$ and (α,Φ) is a compatible pair, that is, for all $g \in G$ and all $x \in \mathcal{X}$, $(g\cdot x)\Phi = (g\alpha)\cdot x\Phi$, in which case an element $(e,g) \in P(G,\mathcal{X},\mathcal{Y})$ is mapped to the element $(e\Phi,g\alpha) \in P(H,\mathcal{U},\mathcal{V})$.

Definition 1.1.5. Let $\varphi\colon S \to T$ be an isomorphism of strongly E^*-unitary inverse semigroups. We say that a mapping $\psi\colon P^*(G,\mathcal{X}^*,\mathcal{Y}^*) \to P^*(H,\mathcal{U}^*,\mathcal{V}^*)$ is a *representation of φ as an isomorphism of P-semigroups* if

(i) $S \simeq P^*(G,\mathcal{X}^*,\mathcal{Y}^*)$ and $T \simeq P^*(H,\mathcal{U}^*,\mathcal{V}^*)$ for some isomorphisms $\pi_S\colon S \to P^*(G,\mathcal{X}^*,\mathcal{Y}^*)$ and $\pi_T\colon T \to P^*(H,\mathcal{U}^*,\mathcal{V}^*)$;

(ii) $\varphi = \pi_S \psi \pi_T^{-1}$;

(iii) ψ is constructed as follows from a compatible pair (α, Φ) consisting of a group isomorphism $\alpha\colon G \to H$ and an order isomorphism $\Phi\colon \mathcal{X} \to \mathcal{U}$ such that $\mathcal{Y}\Phi = \mathcal{V}$: $0\psi = 0$ and $(e,g)\psi = (e\Phi, g\alpha)$, for each $(e,g) \in P^*(G, \mathcal{X}^*, \mathcal{Y}^*) \setminus \{0\}$.

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "\pi_S"'] \arrow[dr, phantom, "\circlearrowright"] & T \arrow[d, "\pi_T"] \\ P^*(G,\mathcal{X}^*,\mathcal{Y}^*) \arrow[r, "\psi"'] & P^*(H,\mathcal{U}^*,\mathcal{V}^*) \end{tikzcd}$$

We now prove that such a representation of an isomorphism of strongly E^*-unitary inverse semigroups always exists.

Proposition 1.1.6. *Let $\varphi\colon S \to T$ be an isomorphism from a strongly E^*-unitary inverse semigroup S to an inverse semigroup T. Then there exists a representation of φ as an isomorphism of P-semigroups.*

Proof. Since T is isomorphic to a strongly E^*-unitary inverse semigroup, then T is itself strongly E^*-unitary, and so there exists a 0-restricted idempotent-pure pre-homomorphism $\lambda_T\colon T \to G^0$. As we saw earlier, λ_T gives rise to a P^*-semigroup $P_T^* = P^*(G, \mathcal{X}_T^*, \mathcal{Y}_T^*)$ isomorphic to T via the mapping $\pi_T\colon T \to P_T^*$ defined by $t\pi_T = ([tt^{-1}, 1], t\lambda_T)$, for all non-zero $t \in T$.

Let $\lambda_S = \varphi\lambda_T$. Then λ_S is a 0-restricted idempotent-pure pre-homomorphism from S into the group with zero G^0 which gives rise, in turn, to a P^*-semigroup $P_S^* = P^*(G, \mathcal{X}_S^*, \mathcal{Y}_S^*)$ isomorphic to S, via the mapping $\pi_S\colon S \to P_S^*$ defined by $s\pi_S = ([ss^{-1}, 1], s\lambda_S)$ for all $s \in S$.

Let $\alpha\colon G \to G$ be the identity map on G and $\Phi\colon \mathcal{X}_S \to \mathcal{X}_T$ be the map defined by $[e,g]\Phi = [e\varphi, g]$, for all $[e,g] \in \mathcal{X}_S$. We claim that (α, Φ) is a compatible pair and that the map $\psi\colon P^*(G, \mathcal{X}_S^*, \mathcal{Y}_S^*) \to P^*(G, \mathcal{X}_T^*, \mathcal{Y}_T^*)$ defined by $0\psi = 0$ and $([e,1],g)\psi = ([e\varphi, 1], g)$, for each non-zero $([e,1],g) \in P^*(G, \mathcal{X}^*, \mathcal{Y}^*)$, is a representation of φ as an isomorphism of P-semigroups.

Let $[e,g], [f,h] \in \mathcal{X}_S$. Then

$$\begin{aligned} [e,g]\Phi \leqslant [f,h]\Phi &\Leftrightarrow [e\varphi, g] \leqslant [f\varphi, h] \\ &\Leftrightarrow \exists\, t \in T \text{ such that } e\varphi = tt^{-1}, f\varphi \leqslant t^{-1}t \text{ and } g^{-1}h \in t\bar{\lambda}_T \\ &\Leftrightarrow \exists\, t \in T \text{ such that } e = (t\varphi^{-1})(t\varphi^{-1})^{-1}, f \leqslant (t\varphi^{-1})^{-1}(t\varphi^{-1}) \\ &\qquad \text{and } g^{-1}h \in t\bar{\lambda}_T\,, \end{aligned}$$

where $\bar{\lambda}_T\colon T \to \mathcal{K}(G)$ is the map that sends 0 to G and $t \neq 0$ to $\{t\lambda_T\}$, so that

$$\begin{aligned} [e,g]\Phi \leqslant [f,h]\Phi &\Leftrightarrow \exists\, s \in S \text{ such that } e = ss^{-1}, f \leqslant s^{-1}s \text{ and } g^{-1}h \in s\varphi\bar{\lambda}_T \\ &\Leftrightarrow \exists\, s \in S \text{ such that } e = ss^{-1}, f \leqslant s^{-1}s \text{ and } g^{-1}h \in s\bar{\lambda}_S \\ &\Leftrightarrow [e,g] \leqslant [f,h]\,, \end{aligned}$$

as $\bar{\lambda}_S \colon S \to \mathcal{K}(G)$ sends 0 to G and $s \neq 0$ to $\{s\lambda_S\}$, that is, $\{s\varphi\lambda_T\}$. It follows that Φ is an order isomorphism. Since $E_S\varphi = E_T$, then $\mathcal{Y}_S\Phi = \mathcal{Y}_T$. Also

$$(h \cdot [e,g])\Phi = [e,hg]\Phi = [e\varphi, hg] = [e\varphi, (hg)\alpha]\,,$$

for all $h \in \mathcal{X}_S$ and $h \in G$, and so (α, Φ) is a compatible pair.

Therefore, by Theorem 1.1.2, the map $\psi' \colon P(G, \mathcal{X}_S, \mathcal{Y}_S) \to P(G, \mathcal{X}_T, \mathcal{Y}_T)$ defined by $([e,1],g)\psi' = ([e,1]\Phi, g\alpha) = ([e\varphi,1],g)$, for all $([e,1],g) \in P(G, \mathcal{X}_S, \mathcal{Y}_S)$, is an isomorphism of P-semigroups. Note that, in particular, $([0,1],g)\psi' = ([0,1],g)$, for all $g \in G$. To get an isomorphism of P^*-semigroups, recall that

- $P^*(G, \mathcal{X}_S^*, \mathcal{Y}_S^*) \simeq P(G, \mathcal{X}_S, \mathcal{Y}_S)/J_S$ with $J_S = \{([0,1],g) \in P(G, \mathcal{X}_S, \mathcal{Y}_S) \colon g \in G\}$ and
- $P^*(G, \mathcal{X}_T^*, \mathcal{Y}_T^*) \simeq P(G, \mathcal{X}_T, \mathcal{Y}_T)/J_T$ with $J_T = \{([0,1],g) \in P(G, \mathcal{X}_T, \mathcal{Y}_T) \colon g \in G\}$.

Moreover, since φ is an isomorphism of semigroups with zero, we have

$$\begin{aligned} J_S\psi' &= \big\{([0,1],g)\psi' \in P(G, \mathcal{X}_T, \mathcal{Y}_T) \colon g \in G\big\} \\ &= \{([0\varphi,1],g) \in P(G, \mathcal{X}_T, \mathcal{Y}_T) \colon g \in G\} \\ &= \{([0,1],g) \in P(G, \mathcal{X}_T, \mathcal{Y}_T) \colon g \in G\} \\ &= J_T\,. \end{aligned}$$

Therefore, we conclude that the map $\psi \colon P^*(G, \mathcal{X}_S^*, \mathcal{Y}_S^*) \to P^*(G, \mathcal{X}_T^*, \mathcal{Y}_T^*)$ defined by $0\psi = 0$ and $([e,1],g)\psi = ([e\varphi,1],g)$, for each non-zero $([e,1],g) \in P^*(G, \mathcal{X}_S^*, \mathcal{Y}_S^*)$, is an isomorphism between P^*-semigroups. Further, we have

$$s\pi_S\psi = ([ss^{-1},1], s\lambda_S)\psi = ([(ss^{-1})\varphi,1], s\lambda_S) = ([s\varphi s\varphi^{-1},1], s\varphi\lambda_T) = s\varphi\pi_T\,,$$

for all $s \in S$, so that $\pi_S\psi = \varphi\pi_T$, as required. □

1.2 On congruences and presentations

In this section, we review some material on presentations, namely presentations of quotients of semigroups, that will be useful in Chapter 6 and whose full treatment cannot be found in the usual reference literature on semigroups. We are specially interested in the following facts: that, for a finitely presented semigroup S, the Rees factor semigroup S/I is finitely presented if and only if I is a finitely generated ideal (Corollary 1.2.12) and that we may obtain a description of a presentation for a Rees factor semigroup S/I in terms of a presentation of S and a generating set for I (Proposition 1.2.13). We also introduce the notion of presentation of a semigroup as a strongly E^*-unitary inverse semigroup with zero that admits a 0-restricted idempotent-pure pre-homomorphism into an abelian group with zero and prove that if the quotient S/ρ, of a strongly E^*-unitary inverse semigroup S

which admits a 0-restricted idempotent-pure pre-homomorphism into an abelian group with zero by a congruence ρ on S, is finitely presented as such, then the congruence ρ is finitely generated (Proposition 1.2.8).

We begin by recalling some notions.

Let S be a semigroup. Let A be an alphabet and $R \subseteq A^+ \times A^+$, where A^+ is the free semigroup on A. We say that the $Sgp\left\langle A \mid R\right\rangle$ is a *semigroup presentation* of S if $S \simeq A^+/\rho$, where ρ is the congruence on A^+ generated by R, and write $S = Sgp\left\langle A \mid R\right\rangle$. We call the elements of A *generating symbols* or *generators* and the elements of R *defining relations* or simply *relations* of the presentation. As usual, we sometimes write $u = v$ instead of (u, v), for $(u, v) \in R$. In case R is the empty set, we usually write $Sgp\left\langle A \mid \right\rangle$ instead of $Sgp\left\langle A \mid \emptyset\right\rangle$.

If S is an inverse semigroup, then $Inv\left\langle A \mid R\right\rangle$ is an *inverse semigroup presentation* of S if A is an alphabet, $A^{-1} = \{a^{-1} \colon a \in A\}$ is an alphabet disjoint from A and in one-one correspondence with A, $R \subseteq \left(A \cup A^{-1}\right)^+ \times \left(A \cup A^{-1}\right)^+$, and $S \simeq \left(A \cup A^{-1}\right)^+/\rho$, where ρ is the congruence on $\left(A \cup A^{-1}\right)^+$ generated by R and by the *Wagner congruence* on $\left(A \cup A^{-1}\right)^+$, that is, the congruence generated by the set

$$\left\{\left(uu^{-1}u, u\right), \left(uu^{-1}vv^{-1}, vv^{-1}uu^{-1}\right) : u, v \in \left(A \cup A^{-1}\right)^+\right\} .$$

Equivalently, $S \simeq FIS_A/\upsilon$, where υ is the congruence on the free inverse semigroup FIS_A on A generated by R. The inverse semigroup defined by the inverse semigroup presentation $Inv\left\langle A \mid R\right\rangle$ is defined by the semigroup presentation

$$Sgp\left\langle A \cup A^{-1} \mid R, uu^{-1}u = u, uu^{-1}vv^{-1} = vv^{-1}uu^{-1} \text{ with } u, v \in \left(A \cup A^{-1}\right)^+\right\rangle$$

(see, for example, [7, Remark 1.4]). As for semigroup presentations, for inverse semigroup presentations we write $Inv\left\langle A \mid \right\rangle$ instead of $Inv\left\langle A \mid \emptyset\right\rangle$.

The difference in considering, for an inverse semigroup S, a semigroup presentation or an inverse semigroup presentation of S is, therefore, that the latter concentrates on the properties of S as an inverse semigroup, rather than as an arbitrary semigroup. For instance, we say that a semigroup is *finitely presented* if it admits a presentation with finite set of generators and finite set of relations, and that it is *infinitely presented* otherwise. Thus, for example, the free inverse semigroup on a finite set A is infinitely presented as a semigroup but finitely presented as an inverse semigroup [57]:

$$\begin{aligned} FIS_A &= Sgp\left\langle A \cup A^{-1} \mid uu^{-1}u = u, uu^{-1}vv^{-1} = vv^{-1}uu^{-1} \text{ with } u, v \in \left(A \cup A^{-1}\right)^+\right\rangle \\ &= Inv\left\langle A \mid \right\rangle . \end{aligned}$$

We can also consider the notion of presentation of a semigroup with zero. If S is a semigroup with zero, then $Sgp_0\left\langle A \mid R\right\rangle$ is a *presentation of a semigroup with zero* for S if $S \simeq A^+/\tau$, where τ is the congruence on A^+ generated by R and all the pairs $(00, 0)$, $(a0, 0)$, $(0a, 0)$, with $a \in A$. Although 0 is a generating symbol of the presentation, we often omit reference to it in A. The notion of *inverse semigroup presentation of a semigroup with zero* is defined analogously and denoted $Inv_0 \langle A \mid R \rangle$.

In Chapter 6 we will come across another notion that is inspired by these concepts. The following, simple, observation will be important:

Remark 1.2.1. If S is a strongly E^*-unitary inverse semigroup which admits a 0-restricted idempotent-pure pre-homomorphism $\lambda \colon S \to G^{\bar{0}}$ into an abelian group with zero, then, for all $x, y \in S$ such that $xyx^{-1}y^{-1} \neq 0$, we have

$$(xyx^{-1}y^{-1})\lambda = x\lambda\, y\lambda\, (x^{-1})\lambda\, (y^{-1})\lambda = x\lambda\, y\lambda\, (x\lambda)^{-1}\, (y\lambda)^{-1} = 1\,,$$

and so $xyx^{-1}y^{-1}$ is an idempotent in S. Since $xyx^{-1}y^{-1}$ is trivially an idempotent when $xyx^{-1}y^{-1} = 0$, then $xyx^{-1}y^{-1}$ is an idempotent in S for all $x, y \in S$. Thus, if $Inv_0 \langle A \mid T \rangle$ is a presentation for S as an inverse semigroup with zero, then all relations of the form $(uvu^{-1}v^{-1})^2 = uvu^{-1}v^{-1}$, with $u, v \in (A \cup A^{-1})^+$, have to be a consequence of the relations in T.

Lemma 1.2.2. *Let S be a strongly E^*-unitary inverse semigroup, generated by A as an inverse semigroup, which admits a 0-restricted idempotent-pure pre-homomorphism into an abelian group with zero. Then there exists a binary relation $R \subseteq (A \cup A^{-1})^+ \times (A \cup A^{-1})^+$ such that $S \simeq (A \cup A^{-1})^+/\omega$ for the congruence ω on $(A \cup A^{-1})^+$ generated by R, the Wagner congruence, all pairs $(00, 0)$, $(a0, 0)$ and $(0a, 0)$ with $a \in A$, and the set*

$$\varepsilon = \left\{ \left(uvu^{-1}v^{-1}\right)^2 = uvu^{-1}v^{-1} \colon u, v \in \left(A \cup A^{-1}\right)^+ \right\}.$$

Proof. Since $xyx^{-1}y^{-1}$ is an idempotent in S for all $x, y \in S$, then S is a homomorphic image of the semigroup

$$F = Inv_0 \left\langle A \mid \left(uvu^{-1}v^{-1}\right)^2 = uvu^{-1}v^{-1}\ \left(u, v \in \left(A \cup A^{-1}\right)^+\right) \right\rangle,$$

by the epimorphism $\varphi \colon F \to S$, say. Thus, $S \simeq F/\ker\varphi$. Since, by definition, F is isomorphic to the quotient of $(A \cup A^{-1})^+$ by the congruence θ on $(A \cup A^{-1})^+$ generated by the Wagner congruence together with all pairs $(a0, 0)$, $(0a, 0)$ and $(00, 0)$, with $a \in A$, and all relations in ε, we have that S is isomorphic to the quotient of $(A \cup A^{-1})^+$ by the congruence on $(A \cup A^{-1})^+$ generated by θ together with $R = \ker\varphi$. ☐

Thus, we may define

Definition 1.2.3. Let S be a strongly E^*-unitary inverse semigroup that admits a 0-restricted idempotent-pure pre-homomorphism into an abelian group with zero.

1. Let A be a set, A^{-1} a set disjoint from A in one-one correspondence with A, and R a binary relation on $(A \cup A^{-1})^+$. We say that $SE^*Inv \langle A \mid R \rangle$ is a *presentation of S as a strongly E^*-unitary inverse semigroup that admits a 0-restricted idempotent-pure*

pre-homomorphism into an abelian group with zero, and write $S = SE^{*}Inv\,\langle A \mid R\rangle$, if $S \simeq (A \cup A^{-1})^{+}/\omega$, where ω is the congruence on $(A \cup A^{-1})^{+}$ generated by R together with the Wagner congruence on $(A \cup A^{-1})^{+}$, all pairs $(00, 0)$, $(a0, 0)$ and $(0a, 0)$, with $a \in A$, and the set $\varepsilon = \left\{ \left(uvu^{-1}v^{-1}\right)^{2} = uvu^{-1}v^{-1} \colon u, v \in \left(A \cup A^{-1}\right)^{+} \right\}$.

2. We say that S is *finitely presented as a strongly E^{*}-unitary inverse semigroup that admits a 0-restricted idempotent-pure pre-homomorphism into an abelian group with zero* if there exists such a presentation for S with finite A and R and that S is *infinitely presented* otherwise.

Remark 1.2.4. Notice that if a strongly E^{*}-unitary inverse semigroup S which admits a 0-restricted idempotent-pure pre-homomorphism into an abelian group with zero is infinitely presented as such, then it is also infinitely presented as an inverse semigroup with zero. In fact, if $S = SE^{*}Inv\,\langle A|R\rangle$ is infinitely presented as a strongly E^{*}-unitary inverse semigroup that admits a 0-restricted idempotent-pure pre-homomorphism into an abelian group with zero and $S = Inv_{0}\,\langle A|T\rangle$ with A and T finite, then, since all relations in $\varepsilon \cup R$ have to be a consequence of the relations in T, we would have that $SE^{*}Inv\,\langle A|T\rangle$ is a finite presentation for S as a strongly E^{*}-unitary inverse semigroup that admits a 0-restricted idempotent-pure pre-homomorphism into an abelian group with zero, a contradiction.

We will now state and prove some results that will be useful in our study of tiling semigroups.

Proposition 1.2.5. *Let S be a finitely generated semigroup and ρ a congruence on S. If S/ρ is finitely presented, then ρ is finitely generated.*

Recall that if S is a semigroup, R is a binary relation on S, and $a, b \in S$, we say that a *is connected to b by an elementary R-transition* if $a = xcy$ and $b = xdy$ for some $x, y \in S^{1}$, where either (c, d) or (d, c) belong to R, and write $a \overset{R}{\to} b$.

Proposition 1.2.6 ([21], Proposition 1.5.9)**.** *Let S be a semigroup, let R be a binary relation on S, and let ρ be the congruence on S generated by R. Then, for all $a, b \in S$, we have $(a, b) \in \rho$ if and only if $a = b$ or there exist a positive integer n and a sequence*

$$a = z_{1} \overset{R}{\to} z_{2} \overset{R}{\to} \dots \overset{R}{\to} z_{n} = b$$

of elementary R-transitions connecting a and b.

We write $a \overset{R}{\rightsquigarrow} b$ for a and b as in the previous result.

For the remainder of this thesis, given a positive integer n, denote the set $\{1, \dots, n\}$ by $[n]$.

Proof of Proposition 1.2.5. Assume that S is a finitely generated, by a set A, say, and that S/ρ is finitely presented. Let $S/\rho = Sgp\,\langle A \mid (z_{1}, w_{1}), \dots, (z_{n}, w_{n})\rangle$ be a finite presentation for

S/ρ (the fact that S/ρ is finitely presented implies that it is finitely presented with respect to any finite set generating set; see, for example, [55]). Thus, S/ρ is isomorphic to the quotient of A^+ by the congruence generated by the pairs (z_i, w_i), with $i \in [n]$.

Let $\delta\colon A^+ \to S$ be the homomorphism from A^+ onto S which takes each $a \in A$ to the corresponding element $a\delta \in S$. Let μ be the congruence on S generated by the finite subset $\{(z_1\delta, w_1\delta), \ldots, (z_n\delta, w_n\delta)\}$ of $S \times S$. We claim that $\rho = \mu$.

For each $j \in [n]$, we have $(z_j\delta, w_j\delta) \in \rho$, as $z_j\delta, w_j\delta \in S$ and S/ρ is isomorphic to the quotient of A^+ by the congruence generated the pairs (z_i, w_i), with $i \in [n]$. Since μ is, by definition, generated by the pairs $(z_i\delta, w_i\delta)$, with $i \in [n]$, we conclude that $\mu \subseteq \rho$.

Conversely, let $x, y \in S$ be such that $(x, y) \in \rho$. Let $u, v \in A^+$ be such that $u\delta = x$ and $v\delta = y$. Then $u\delta\rho^\natural = v\delta\rho^\natural$, and so there is a finite sequence of elementary transitions from u to v using the relations from the presentation of S/ρ. Therefore, the images under δ of the elements from the sequence give a finite sequence of elementary transitions, of μ, from $u\delta$ to $v\delta$. In fact, if $u' \to v'$, with $u', v' \in A^+$, is an elementary transition using a relation from the presentation of S/ρ then by definition there exist $r, s \in A^*$ such that either $u' = rz_is$ and $v' = rw_is$ or $u' = rw_is$ and $v' = rz_is$, for some pair (z_i, w_i). For instance in the first case (the second case is analogous), $u'\delta = (rz_is)\delta = r\delta z_i\delta s\delta$ and $v'\delta = (rw_is)\delta = r\delta w_i\delta s\delta$, with $r\delta, s\delta \in S^1$, so that $u'\delta \to v'\delta$ is an elementary transition using the relation $(z_i\delta, w_i\delta)$ from μ. Thus, $(x, y) \in \mu$, and we conclude that $\rho \subseteq \mu$.

Hence, $\rho = \mu$, and so ρ is finitely generated. □

Analogous results hold for inverse semigroups and for strongly E^*-unitary inverse semigroups which admit a 0-restricted idempotent-pure pre-homomorphism into an abelian group with zero:

Proposition 1.2.7. *Let S be a finitely generated inverse semigroup and ρ a congruence on S such that S/ρ is finitely presented as an inverse semigroup. Then ρ is finitely generated.*

Proof. Since the proof is similar to that of Proposition 1.2.5, we shall only give a sketch of it.

Suppose S is generated, as an inverse semigroup, by a finite set A and that $\mathit{Inv}\,\langle A \mid (z_1, w_1), \ldots, (z_n, w_n)\rangle$ is a finite inverse semigroup presentation for S/ρ. Then S is generated, as a semigroup, by the finite set $A \cup A^{-1}$ and S/ρ is isomorphic to the quotient of $(A \cup A^{-1})^+$ by the congruence generated by the pairs (z_i, w_i), with $i \in [n]$, together with the Wagner congruence. Let $\delta\colon (A \cup A^{-1})^+ \to S$ be the homomorphism from $(A \cup A^{-1})^+$ onto S such that $x\delta = x$, for each $x \in A \cup A^{-1}$, and let μ be the congruence on S generated by the finite subset $\{(z_1\delta, w_1\delta), \ldots, (z_n\delta, w_n\delta)\}$ of $S \times S$.

That $\mu \subseteq \rho$ can be shown as in the proof of Proposition 1.2.5. To show the converse inclusion, note that, under δ, a transition using a relation from the Wagner congruence turns into an equality. For instance, if $u', v' \in (A \cup A^{-1})^+$ are such that $u' = rts$ and $v' = rtt^{-1}ts$ for some $r, s \in (A \cup A^{-1})^*$ and $t \in (A \cup A^{-1})^+$, then $v'\delta = r\delta t\delta(t\delta)^{-1}t\delta s\delta = r\delta t\delta s\delta = u'\delta$, since δ is a homomorphism and S is an inverse semigroup. Consequently, a finite sequence of elementary transitions using the relations from the set $\{(z_i, w_i) : i \in [n]\}$ or from the Wagner congruence yields, under δ, a finite sequence of elementary transitions using the relations from the set $\{(z_i\delta, w_i\delta) : i \in [n]\}$.

We thus have the desired conclusion. $\square$

Proposition 1.2.8. *Let S be a finitely generated strongly E^*-unitary inverse semigroup which admits a 0-restricted idempotent-pure pre-homomorphism into an abelian group with zero and ρ a congruence on S. If S/ρ is finitely presented as a strongly E^*-unitary inverse semigroup which admits a 0-restricted idempotent-pure pre-homomorphism into an abelian group with zero, then ρ is finitely generated.*

Proof. We shall not give a detailed proof of this result as it is entirely analogous to that of the previous one — here, both the transitions using pairs from the Wagner congruence on $(A \cup A^{-1})^+$ and from the set ε become equalities under the map $\delta\colon (A \cup A^{-1})^+ \to S$. $\square$

For the converse of Proposition 1.2.5, stronger conditions are needed.

Proposition 1.2.9. *Let S be a finitely presented semigroup and ρ a congruence on S. If ρ is finitely generated, then S/ρ is finitely presented.*

Recall the useful fact:

Proposition 1.2.10 ([21], Theorem 1.5.4)**.** *Let S be a semigroup and σ and τ congruences on S such that $\tau \subseteq \sigma$. Then*

$$\sigma/\tau = \left\{ \left(x\tau^\natural, y\tau^\natural \right) \in S/\tau \times S/\tau \colon (x, y) \in \sigma \right\}$$

is a congruence on S/τ and $(S/\tau) / (\sigma/\tau) \cong S/\sigma$.

Proof of Proposition 1.2.9. Let $S = Sgp\langle A \mid T \rangle$ be a finite semigroup presentation for S and let τ be the congruence on A^+ generated by T, so that $S \simeq A^+/\tau$ via, say, the isomorphism $\phi\colon A^+/\tau \to S$.

Consider the binary relation σ on A^+ defined by: for $u, v \in A^+$,

$$(u, v) \in \sigma \Leftrightarrow (u\tau^\natural\phi, v\tau^\natural\phi) \in \rho\,.$$

Using the fact that τ is a congruence on A^+, that ρ a congruence on $S \simeq A^+/\tau$ and that ϕ is an isomorphism, it is easy to check that σ is a congruence on A^+. For example, to check that σ is compatible with multiplication on the right, let $u, v, w \in A^+$ with $(u, v) \in \sigma$. Then

$$\begin{aligned}
(u, v) \in \sigma &\Leftrightarrow (u\tau^\natural\phi, v\tau^\natural\phi) \in \rho && \text{(by definition of } \sigma)\\
&\Rightarrow (u\tau^\natural\phi w\tau^\natural\phi, v\tau^\natural\phi w\tau^\natural\phi) \in \rho && \text{(since } \rho \text{ is right compatible)}\\
&\Rightarrow ((uw)\tau^\natural\phi, (vw)\tau^\natural\phi) \in \rho && \text{(since } \tau^\natural\phi \text{ is a homomorphism)}\\
&\Rightarrow (uw, vw) \in \sigma && \text{(again by definition of } \sigma).
\end{aligned}$$

It is also easy to show that $\tau \subseteq \sigma$. In fact, for all $u, v \in A^+$ such that $(u, v) \in \tau$, we have $u\tau^\natural = v\tau^\natural$, and so $u\tau^\natural\phi = v\tau^\natural\phi$. Thus, $(u\tau^\natural\phi, v\tau^\natural\phi) \in \rho$ since ρ is reflexive. Then, by definition, $(u, v) \in \sigma$.

Since $\phi\colon A^+/\tau \to S$ is an isomorphism and ρ a congruence on S, then the binary relation $\bar{\rho}$ on A^+/τ defined by: for all $u, v \in A^+$,

$$(u\tau^\natural, v\tau^\natural) \in \bar{\rho} \Leftrightarrow (u\tau^\natural\phi, v\tau^\natural\phi) \in \rho$$

is a congruence on A^+/τ and $S/\rho \simeq (A^+/\tau)/\bar{\rho}$. On the other hand, by Proposition 1.2.10, $A^+/\sigma \simeq (A^+/\tau)/(\sigma/\tau)$, and so we have

$$A^+/\sigma \simeq (A^+/\tau)/(\sigma/\tau) \simeq (A^+/\tau)/\bar{\rho} \simeq S/\rho\,,$$

as, for all $u, v \in A^+$,

$$(u\tau^\natural, v\tau^\natural) \in \sigma/\tau \Leftrightarrow (u, v) \in \sigma \Leftrightarrow (u\tau^\natural\phi, v\tau^\natural\phi) \in \rho \Leftrightarrow (u\tau^\natural, v\tau^\natural) \in \bar{\rho}\,.$$

We claim that σ is finitely generated. Assume that ρ is generated by the finite set $R = \{(s_1, t_1), (s_2, t_2), \ldots, (s_n, t_n)\}$. For each (s_i, t_i) in R, take $(u_i, v_i) \in A^+ \times A^+$ such that $u_i\tau^\natural\phi = s_i$ and $v_i\tau^\natural\phi = t_i$, and consider the set of all such pairs (u_i, v_i), that is,

$$\tilde{R} = \left\{(u_i, v_i) \in A^+ \times A^+ \colon i \in [n]\right\}\,.$$

Notice that $T \cup \tilde{R} \subseteq \sigma$, as $T \subseteq \tau \subseteq \sigma$ and $R \subseteq \rho$ implies that $\tilde{R} \subseteq \sigma$. We claim that σ is generated by $T \cup \tilde{R}$. Let $u, v \in A^+$ with $(u, v) \in \sigma$. Then $(u\tau^\natural, v\tau^\natural) \in \bar{\rho}$ and, since ρ is generated by R, we have $u\tau^\natural \overset{R}{\leadsto} v\tau^\natural$ where each elementary R-transition in this sequence is of the form $z\tau^\natural \overset{R}{\to} z'\tau^\natural$, with $z, z' \in A^+$. Expanding out, we get $z \overset{\tilde{R}\cup T}{\leadsto} z'$, so that $u \overset{\tilde{R}\cup T}{\leadsto} v$. Hence, σ is generated by $T \cup \tilde{R}$.

Since $T \cup \tilde{R}$ is a finite set and $S/\rho \simeq A^+/\sigma$, we conclude that $Sgp\left\langle A \,|\, T \cup \tilde{R}\right\rangle$ is a finite presentation for S/ρ. ☐

In order to draw conclusions about the finite or infinite presentability of Rees factor semigroups, we first show that

Lemma 1.2.11. *If S is a finitely generated semigroup and I a proper ideal of S, then the Rees congruence ρ_I is finitely generated if and only if I is finitely generated as an ideal.*

Proof. Let S be a finitely presented semigroup and I a proper ideal of S. Recall that ρ_I is defined by: for all $x, y \in S$,

$$(x, y) \in \rho_I \;\Leftrightarrow\; x = y \text{ or } x, y \in I\,.$$

Suppose I is finitely generated as an ideal, that is, $I = S^1a_1S^1 \cup \ldots \cup S^1a_mS^1$ for some $a_1, \ldots, a_m \in S$. Let $X = \{x_1, \ldots, x_n\}$ be a generating set for S. We will show that the (finite) set $R = \{(a_1, a_i), (x_j a_1, a_1), (a_1 x_j, a_1) \colon i \in [m], j \in [n]\}$ generates ρ_I. Let $(x, y) \in \rho_I$, with $x \neq y$. Then, by definition of ρ_I, we have $x, y \in I$, and so $x = r a_k s$ and $y = r' a_l s'$, for some $k, l \in [m]$ and $r, s, r', s' \in S$. Then

$$x = r a_k s \overset{R}{\to} r a_1 s \overset{R}{\rightsquigarrow} a_1 s \overset{R}{\rightsquigarrow} a_1 \overset{R}{\rightsquigarrow} r' a_1 \overset{R}{\rightsquigarrow} r' a_1 s' \overset{R}{\to} r' a_l s' = y \,,$$

since r, s, r' and s' belong to S and S is finitely generated by the elements in X. In fact, if, for example, $r = x_{j_1} x_{j_2} \ldots x_{j_p}$, with $x_{j_1}, x_{j_2}, \ldots x_{j_p} \in X$, then

$$r a_1 s = x_{j_1} x_{j_2} \ldots x_{j_p} a_1 s \overset{R}{\to} x_{j_1} x_{j_2} \ldots x_{j_{p-1}} a_1 s \overset{R}{\to} \ldots \overset{R}{\to} x_{j_1} a_1 s \overset{R}{\to} a_1 s \,.$$

Thus, there is a finite sequence of elementary R-transitions connecting x and y. Therefore, R generates ρ_I and, hence, ρ_I is finitely generated.

Conversely, suppose that ρ_I is finitely generated, by a set $R = \{(u_i, v_i), (v_i, u_i) \colon i \in [n]\}$, say. Without loss of generality, we may assume that $u_i \neq v_i$ for all $i \in [n]$. We claim that the set $A = \{u_i \colon i \in [n]\}$ generates I as an ideal of S.

On the one hand, since $u_i \neq v_i$ for all $i \in [n]$ and $(u_i, v_i) \in R$ implies that $(u_i, v_i) \in \rho_I$, then $R \subset I \times I$. So, if $u_i \in A$, then $u_i \in I$, so that $S^1 A S^1 \subseteq I$. On the other hand, if $u \in I$, then either $u = u_i$ for some $i \in [n]$, in which case $u \in S^1 A S^1$, or $u \neq u_i$ for all $i \in [n]$. Let $u_i \in A$. Since $(u, u_i) \in \rho_I$, there exist a positive integer m and a sequence

$$u = z_1 \overset{R}{\to} z_2 \overset{R}{\to} \ldots \overset{R}{\to} z_m = u_i$$

of elementary R-transitions connecting u and u_i. In particular, $u = z_1 = x u_j y$ for some $x, y \in S^1$ and $j \in [n]$. Thus, $u \in S^1 A S^1$. Therefore, $I \subseteq S^1 A S^1$. We conclude that $I = S^1 A S^1$, and, hence, that I is finitely generated as an ideal. □

In view of the previous results, the following is now immediate:

Corollary 1.2.12. *Let S be a finitely presented semigroup and I an ideal in S. Then S/I is finitely presented if and only if I is finitely generated as an ideal.*

Proof. Since $S/I = S/\rho_I$, we have that S/I is finitely presented if and only if S/ρ_I is finitely presented. By Propositions 1.2.5 and 1.2.9, we have that S/ρ_I is finitely presented if and only if ρ_I is finitely generated; by Lemma 1.2.11, ρ_I is finitely generated if and only if I is finitely generated as an ideal of S. □

The following result, which also shows the converse of the previous corollary, will be important on its own:

Proposition 1.2.13. *Let $S = \langle A \mid T \rangle$ be a semigroup and I an ideal of S. If X is a generating set for I as an ideal, then*

$$S/I = \langle A \mid T, x = 0 \ (x \in X) \rangle \ .$$

Proof. Let τ be the congruence on A^+ generated by T; then $S \simeq A^+/\tau$ via the isomorphism $\phi\colon A^+/\tau \to S$, say. Let $X = \{x_j\colon j \in J\}$ and, for each $x_j \in X$, take $u_j \in A^+$ such that $x_j = u_j\tau^\natural\phi$. We claim that $S/I = \langle A \mid T, u_j = 0\ (j \in J)\rangle$.

Let μ be the congruence on A^+ generated by $T \cup \{u_j = 0\colon j \in J\}$.

$$\begin{tikzcd}
A^+ \arrow[r, two heads, "\tau^\natural"] \arrow[d, two heads, "\mu^\natural"'] & A^+/\tau \overset{\phi}{\simeq} S \arrow[r, two heads, "\rho_I^\natural"] & S/I \\
A^+/\mu \arrow[urr, dashed, two heads, "\psi"'] & &
\end{tikzcd}$$

To guarantee that the composition $\tau^\natural\rho_I^\natural$ (in fact, $\tau^\natural\phi\rho_I^\natural$, to be precise, but we omit ϕ for simplicity) factors through $\mu^\natural$, we show that $\mu \subseteq \ker(\tau^\natural\rho_I^\natural)$. So let $(u, v) \in \mu$. If $u = v$, then trivially $u\tau^\natural\rho_I^\natural = v\tau^\natural\rho_I^\natural$; if $u \neq v$, then there exists a finite sequence of transitions $u = z_1 \to z_2 \to \ldots \to z_m = v$ using pairs from $T \cup \{(u_j, 0)\colon j \in J\}$. Since a pair from T turns into equality under $\tau^\natural$ and a pair of the form $(u_j, 0)$ turns into equality under $\rho_I^\natural$, we have $u\tau^\natural\rho_I^\natural = v\tau^\natural\rho_I^\natural$. Then there exists a homomorphism $\psi\colon A^+/\mu \to S/I$ such that $\mu^\natural\psi = \tau^\natural\rho_I^\natural$, which is onto since $\tau^\natural\rho_I^\natural$ is onto.

It remains to prove that ψ is injective. Let $u, v \in A^+$ and suppose that $(u\mu)\psi = (v\mu)\psi$. Then $u\tau^\natural\rho_I^\natural = v\tau^\natural\rho_I^\natural$. Thus, $u\tau^\natural = v\tau^\natural$ or $u\tau^\natural, v\tau^\natural \in I$, by definition of Rees congruence. In the first case, $u\mu^\natural = v\mu^\natural$ as $\tau \subseteq \mu$; in the second case, $u\mu^\natural = v\mu^\natural$ as $\{u_j = 0\colon j \in J\} \subseteq \mu$ and $X = \{u_j\tau\colon j \in J\}$ generates I.

Hence, $S/I \simeq A^+/\mu$ and so $S/I = \langle A \mid T, u_j = 0\ (j \in J)\rangle$. $\square$

1.3 On languages

We will pay special attention to one-dimensional tilings. Because of this, some concepts from language theory will arise naturally. In particular, the notions of factorial language, language of factors of a bi-infinite word and of minimal forbidden word of a language will be important. In this section, we review these notions and a few related ones that help to put them into context.

Let Σ be a non-empty set. We denote by Σ^* the free monoid on Σ (and by Σ^+ the free semigroup on Σ, as mentioned in Section 1.2); thus, $\Sigma^* = \Sigma^+ \cup \{\epsilon\}$, where ϵ is the empty word. A *language over* Σ is any subset of Σ^*.

Contrary to the spirit of this chapter, we shall mention two concepts, those of *regular* and *context-free languages*, without giving their proper definition, in order to avoid long definitions that would serve no other purpose. These can be found, for example, in [1] or [20]. For just a flavour, we recall that, whereas regular languages are recognized by finite state automata, context-free languages require pushdown automata, a device that allows for some kind of memory.

The following is a pumping lemma for context-free languages, and we will make use of it. Recall that if w is a word over the alphabet Σ, we define the *length* $|w|$ *of* w *as* $|w| = 0$ if $w = \epsilon$ and $|w| = m$ if $w = a_1 \dots a_m$ for some $m \geqslant 1$ with $a_1, \dots, a_m \in \Sigma$.

Lemma 1.3.1 ([20], Theorem 5.5.1)**.** *If $L \subseteq \Sigma^*$ is a context-free language, then there is a constant k such that for each $w \in L$ with $|w| > k$ there exist $x, y, z, u, v \in \Sigma^*$ such that $w = xuyvz$ and*

1) $|uv| \geqslant 1$,

2) $|uyv| < k$,

3) $xu^m yv^m z \in L$, for all $m \geqslant 0$.

Let $u, v \in \Sigma^*$. We say that u is a *factor* of v if there exist $x, y \in \Sigma^*$ such that $v = xuy$ and that u is a *proper factor* of v if there exist $x, y \in \Sigma^*$ not both the empty word such that $v = xuy$. Given a word u (respectively, a language L), we denote by $F(u)$ (respectively, $F(L)$) the set of factors of u (respectively, the set of factors of words in L). We say that a language L is *factorial* if every factor of every word in L belongs to L; that is, if $F(L) = L$.

A *bi-infinite word* over an alphabet Σ is an element of $\Sigma^{\mathbb{Z}}$, that is, a map from $\mathbb{Z}$ to Σ. It will be convenient to think of a bi-infinite word as a sequence, infinite both to the left and to the right, of elements from Σ:

$$\dots a_{-3}\, a_{-2}\, a_{-1}\, a_0\, a_1\, a_2\, a_3 \dots ,$$

with $a_i \in \Sigma$ for each $i \in \mathbb{Z}$. In order to avoid trivialities, we shall identify bi-infinite words under translations in $\mathbb{Z}$ (or shifts). Thus, if $\alpha, \beta \in \Sigma^{\mathbb{Z}}$ are such that there exists $k \in \mathbb{Z}$ with $\alpha(n) = \beta(n + k)$ for all $n \in \mathbb{Z}$, then α and β are the same bi-infinite word.

If α is a bi-infinite word over an alphabet Σ, we denote by $F(\alpha)$ the set of all finite factors of α. Then $F(\alpha)$ is a language over Σ, which is called the *language of factors of the bi-infinite word* α. We denote by $\mathcal{F}_{biw}$ the class of languages consisting of the finite factors of a bi-infinite word over some alphabet. It is very easy to see that if $L \in \mathcal{F}_{biw}$, then

1) L is factorial;

2) L is *(bi-)extensible*, in the sense that for each $u \in L$ there exist $a, b \in \Sigma$ such that $aub \in L$;

3) for all $u, v \in L$ there exists $w \in L$ such that $u, v \in F(w)$.[1]

It will be useful in the sequel to note that a factorial language need not satisfy either condition 2 or condition 3. In fact,

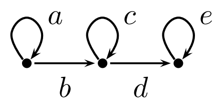
[1]In fact, it is claimed in [3] that this set of properties characterizes the class $\mathcal{F}_{biw}$.

Example 1.3.2. Consider the language over $\Sigma = \{a, b\}$

$$L = \{a, b, ba, baa, aa, baaa, aaa, baaaa, aaaa, \ldots\} = a^* \cup ba^* .$$

Then L is factorial and satisfies condition 3, but it does not satisfy condition 2, since $b \in L$ but neither ab nor bb belong to L (so that there do not exist $x, y \in \Sigma$ such that $xby \in L$).

Example 1.3.3. Consider the language L over $\Sigma = \{a, b, c, d, e\}$ recognized by the automaton



in which all states are initial and final. Then L is factorial and extensible, but there exists no word in L which has both bcd and $bccd$ as factors. Therefore, L does not satisfy condition 3.

Some special classes of languages are immediately seen to be included in $\mathcal{F}_{biw}$. For example:

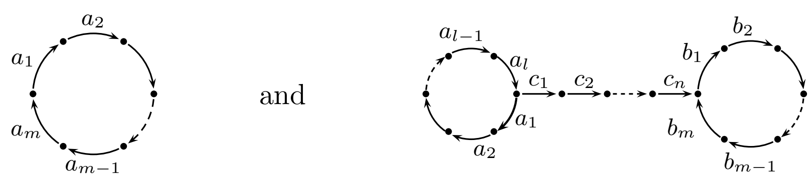
- *periodic languages*, that is, languages of the form $F(u^*)$ for some $u \in \Sigma^+$. The bi-infinite word associated to $F(u^*)$ is

$$\ldots u\,u\,u\,u\,u\,u\,u \ldots .$$

- *(two-way) ultimately periodic languages*, that is, languages of the form $F(u^*wv^*)$ for some $u, v \in \Sigma^+$ and $w \in \Sigma^*$. The bi-infinite word associated to $F(u^*wv^*)$ is

$$\ldots u\,u\,u\,u\,w\,v\,v\,v \ldots .$$

Evidently, periodic and ultimately periodic languages are always regular languages, as they are recognized by the folowing finite automata, respectively,

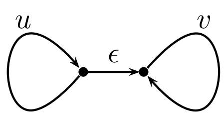


(where $u = a_1 \cdots a_m$)

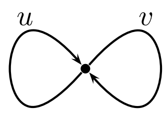
(where $u = a_1 \cdots a_l$, $v = b_1 \cdots b_m$, and $w = c_1 \cdots c_n$)

Figure 1.1: Automata for periodic and ultimately periodic languages

where all states are both initial and final. Notice that, when w is the empty word, $F(u^*v^*)$ is recognized by the automaton



but not by the automaton



Also note that, however, periodic and ultimately periodic languages do not exhaust the class of regular languages of factors of a bi-infinite word.

Example 1.3.4. Let Σ be a finite alphabet and consider the language Σ^+. As shown in [58] (cf. Example 2.3.8), Σ^+ is the language of factors of a bi-infinite word. In fact, it is the language of factors of infinitely many bi-infinite words. Note that, unless $|\Sigma| = 1$, the language Σ^+ cannot be recognized by an automaton of the form of one from Figure 1.1.

Furthermore, the language of factors of a bi-infinite word needs not be regular; in fact, the next example shows that it needs not even be context-free.

Example 1.3.5. Consider the bi-infinite word α:

$$\cdots a\,b\,b\,b\,b\;a\,b\,b\,b\;a\,b\,b\;a\,b\;a\,a\,b\;a\,a\,a\,b\;a\,a\,a\,a\,b\cdots$$

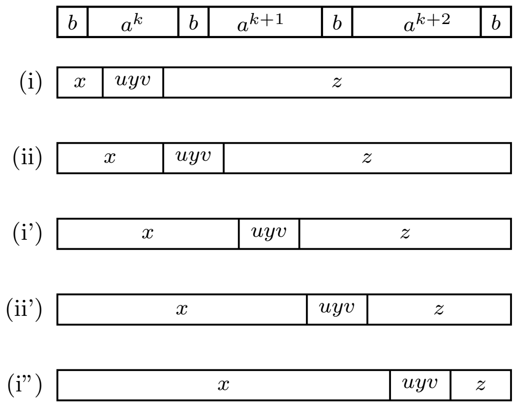
and let L be the language of factors of α. We claim that L is not context-free.

In order to obtain a contradiction, suppose that L is context-free and let k be as in Lemma 1.3.1. Consider the word $w = ba^kba^{k+1}ba^{k+2}b \in L$. Then the fact that $|w| > k$ implies that $w = xuyvz$, for some $x, y, z, u, v \in \{a, b\}^*$, such that $|uv| \geqslant 1$, $|uyv| < k$ and $xu^myv^mz \in L$, for all $m \geqslant 0$.

First, notice that neither u nor v can contain the letter b. In fact, if for example u contained an occurrence of the letter b, then $u = a^rba^s$ for some $r, s \geqslant 0$, since $|uyv| < k$ implies, in particular, that $|u| < k$, and no factor of w of length less than k can contain more than one b. But then, since xu^myv^mz belongs to L for all $m \geqslant 0$ and L is factorial, we have that $u^m \in L$ for all $m \geqslant 0$. In particular, $u^3 = a^rba^sa^rba^sa^rba^s \in L$, which is impossible since $ua^rba^sa^rba^sa^rba^s$ contains two consequent occurrences of b at the same distance. Therefore, neither u nor v can contain the letter b, and so both are powers of a.

Thus, for each $m \geqslant 0$, there exist $p_m, q_m, r_m \geqslant k$ (depending on m) such that $xu^myv^mz = ba^{p_m}ba^{q_m}ba^{r_m}b$, as there are no repetitions of the letter b. Moreover, as both the first and the last letter of xu^myv^mz is a b and xu^myv^mz contains four separate b's, then, for all $m \geqslant 0$, we have that $xu^myv^mz \in L$ if and only if $xu^myv^mz = ba^{n_m}ba^{n_m+1}ba^{n_m+2}b$ for some $n_m \geqslant k$. However, the fact that $|uyv| < k$ and the fact that both u and v are powers of

a imply that there are only five possibilities for the location of the factor uyv on w:



That is: (i) uyv is a factor of a^k; (ii) uyv is a factor of a^kba^{k+1} with b a factor of y; (i') uyv is a factor of a^{k+1}; (ii') uyv is a factor of $a^{k+1}ba^{k+2}$ with b a factor of y; (i") uyv is a factor of a^{k+2}. It is easy to see that, in any case, xu^myv^mz will not be of the form $ba^{n_m}ba^{n_m+1}ba^{n_m+2}b$ for sufficiently large m, as p_m, q_m and r_m need not match with n_m, n_m+1 and n_m+2, respectively. Therefore, xu^myv^mz will not belong to L, a contradiction. We conclude that L is not a context-free language (nor, in particular, a regular language).

Finally, we recall a notion that will play an important role in Chapter 6. Let L be a factorial language. A word $w \in \Sigma^*$ is a *minimal forbidden word for* L if $w \notin L$ but all *proper factors* of w, that is, the factors of w that are neither ϵ nor w, belong to L. We denote the *set of minimal forbidden words for* L by $M(L)$.

Remark 1.3.6. Let L be a factorial language over an alphabet Σ. Suppose that L contains every letter from Σ. Then, if a word $w = a_1 \dots a_m \in \Sigma^+$ is a minimal forbidden word for L, we must have $m \geqslant 2$. Also notice that, if a word w does not belong to L but both its longest proper factors do, then w is a minimal forbidden word for L, since we are assuming that L is factorial. Therefore,

$$M(L) = \{w \in \Sigma^* \setminus L\colon\, w = axb,\ a, b \in \Sigma,\ x \in \Sigma^*,\ \text{and } ax, xb \in L\}\ .$$

Also note that, since L is factorial, then $\Sigma^* \setminus L$ is an ideal of Σ^* and that $M(L)$ is the unique minimal generating set of $\Sigma^* \setminus L$.

Chapter 2

Generalized Bruck-Reilly extensions

The resemblance between the operations defined on the Bruck-Reilly extension of a group determined by an endomorphism of the group and on a certain kind of tiling semigroup (associated with a type of tiling which we will introduce in Chapter 3 and call hypercubic tiling), led us to investigate their relationship. In the absence of an existing construction that would include both cases, we looked for a new kind of semigroup, namely the generalized Bruck-Reilly extension of a semigroup by an anti-homomorphism. As we will see, this construction generalizes the Bruck-Reilly extension and every tiling semigroup associated to a hypercubic tiling is isomorphic to a Rees factor semigroup of a subsemigroup of this new construction.

This chapter is devoted to the description of such semigroups, along with some background material specific to the topic. Thus, in Section 2.1 we review ordered groups of left quotients, an important underlying ingredient in our extensions; in Section 2.2 we recall some theory on bisimple inverse semigroups, namely Clifford's and Reilly's contributions [8, 54] and the Bruck-Reilly extension of a group; finally, Section 2.3 consists of the construction and study of the generalized Bruck-Reilly extension.

The application of this extension to tiling semigroups will be postponed to Section 4.3. The reason for the construction being placed here even before the definition of tiling semigroup has to do with its fairly general nature.

2.1 Preliminaries: the partially ordered group of left quotients

In this section, we review some aspects concerning groups of left quotients, ordered groups, and lattice ordered groups, which will be useful not only in Section 2.3 but also in Section 4.3. The main focus will be on partially ordered groups of left quotients, which seem to possess the minimal conditions for our new construction.

Let S be a semigroup. A group G is a *group of left quotients* of S if S is a subsemigroup of G and every element of G can be written in the form $a^{-1}b$ for some $a, b \in S$. It has been shown by Dubreil [15] that a cancellative semigroup S is embeddable in a group of left quotients of S if and only if S is *right reversible*, that is, if $Sa \cap Sb \neq \emptyset$ for all $a, b \in S$.

In a group of left quotients of a right reversible cancellative semigroup S, we have [9]:

(i) $a^{-1}b = c^{-1}d \Leftrightarrow (\forall\, x, y \in S \quad xa = yc \Leftrightarrow xb = yd)$;

(ii) $(a^{-1}b)(c^{-1}d) = (xa)^{-1}(yd)$, where $x, y \in S$ are any elements such that $xb = yc$.

Let R be a monoid. Recall that an element $u \in R$ is a *right unit* of R if it has a right inverse in R, that is, if there exists $v \in R$ such that $uv = 1$. *Left units* are defined dually, and an element is a *unit* of R it if is both a right and a left unit.

Lemma 2.1.1. *Let R be a right reversible cancellative monoid and G a group of left quotients of R. For each g and h in G, we have $Rg = Rh$ if and only if $h = ug$ for some unit u of R.*

Proof. Let $g, h \in G$. Suppose that $Rg = Rh$. Then $h \in Rg$ and $g \in Rh$, so that $h = ug$ and $g = vh$ for some $u, v \in R$. Thus, $h = ug = uvh$ and so $uv = 1$. As $vu = 1$ as well, we conclude that u and v are units of R. Conversely, suppose we have $h = ug$ with u a unit of R. Then $h \in Rg$, which implies that $Rh \subseteq Rg$. From the fact that u is a unit, we have $vu = 1$ for some $v \in R$. Thus, $g = 1g = vug = vh \in Rh$, so that $Rg \subseteq Rh$. Therefore, $Rg = Rh$. □

We note that the previous result is in fact true for $g, h \in R$ (instead of $g, h \in G$) in a cancellative monoid R, but the formulation presented will be the most useful.

Given a right reversible cancellative monoid R and a group G of left quotients of R, consider the following binary relation on G:

$$g \leqslant h \Leftrightarrow h \in Rg\,. \tag{2.1}$$

Lemma 2.1.2. *Let R be a right reversible cancellative monoid with no non-trivial units and G a group of left quotients of R. Then the binary relation $\leqslant$ defines a right compatible partial order on G.*

Proof. Let $g \in G$. Since R is a monoid, we have $g = 1g \in Rg$. Thus, $g \leqslant g$ and we have that $\leqslant$ is reflexive. Let $g, h \in G$ and assume that $g \leqslant h$ and $h \leqslant g$. Then $h \in Rg$ and $g \in Rh$, so that $Rg = Rh$. But then, by Lemma 2.1.1, $h = ug$ for some unit u of R. Since, by assumption, R has no non-trivial units, then $u = 1$ and hence $g = h$. Therefore, $\leqslant$ is anti-symmetric. Now, let $g, h, k \in G$ be such that $g \leqslant h$ and $h \leqslant k$. Then $h = ag$ and $k = bh$ for some $a, b \in R$, so that $k = (ba)g \in Rg$, that is, $g \leqslant k$. Hence, $\leqslant$ is transitive and we conclude that $\leqslant$ is a partial order on G.

To check the right compatibility, let $g, h, k \in G$ be such that $g \leqslant h$. By definition, $h \in Rg$, so that $hk \in Rgk$. Thus, $gk \leqslant hk$, as required. □

For a group G equipped with a partial order $\leqslant$, we define the *positive cone* of G as the set G^+ of all elements greater or equal to the identity element; that is,

$$G^+ = \{g \in G\colon g \geqslant 1\}\ .$$

Next, we prove some properties of the partial order defined by (2.1) that will be useful in the sequel.

For a monoid R, consider the following condition:

(LI) the set of principal left ideals of R forms a semilattice under intersection, that is, for all $a, b \in R$ there exists $c \in R$ such that $Ra \cap Rb = Rc$.

Lemma 2.1.3. *Let R be a cancellative monoid with no non-trivial units satisfying condition (LI), G a group of left quotients of R and $\leqslant$ the right compatible partial order defined on G by (2.1). Then:*

(i) *R is the positive cone of G with respect to $\leqslant$.*

(ii) *for all $a, b \in R$, the least upper bound $a \vee b$ of a and b under $\leqslant$ exists in R.*

Proof. (i) Let $a \in R$. Since $R = R1$, we have $1 \leqslant a$. Thus, $R \subseteq G^+$. Conversely, suppose $g \in G^+$. Then, by definition, $g \in R1 = R$, and so $G^+ \subseteq R$. Therefore, $R = G^+$.

(ii) Let $a, b \in R$. By assumption, there exists (a unique, by Lemma 2.1.1) $c \in R$ such that $Ra \cap Rb = Rc$. Then $Rc \subseteq Ra, Rb$, and so $a, b \leqslant c$. Suppose $d \in R$ is such that $a, b \leqslant d$. Then $d \in Ra \cap Rb = Rc$ and, thus, $c \leqslant d$. Hence, $c = a \vee b$.

□

In particular, we have from part (ii) of the previous lemma that $Ra \cap Rb = R(a \vee b)$, with $a \vee b \in R$ the least upper bound of a and b under $\leqslant$. Further, since $a \vee b \in Ra$ and $a \vee b \in Rb$, then $a \vee b = (b * a)a$ and $a \vee b = (a * b)b$, for some $a * b, b * a \in R$. So, $Ra \cap Rb = R(a * b)b = R(b * a)a$. Notice that, in view of the cancellativity of R, for all $a, b \in R$, the elements $a * b$ and $b * a$ are unique.

It will be extremely useful in what follows to recall the following facts:

Lemma 2.1.4. *In a group G of left quotients of a cancellative monoid R with no non-trivial units which satisfies condition (LI), we have $a * b = (a \vee b)b^{-1}$, for all $a, b \in R$. In particular, we have*

(i) $a * a = 1\,;$ *(ii)* $a * 1 = a\,;$ *(iii)* $1 * a = 1\,,$

for each $a \in R$.

Proof. Properties (i), (ii) and (iii) can be derived either from $a * b = (a \vee b)b^{-1}$ or from the definition of $*$ and the fact that R is cancellative and $a \geqslant 1$ for all $a \in R$. □

Lemma 2.1.5. *Let R be a cancellative monoid with no non-trivial units satisfying condition (LI) and G a group of left quotients of R. Let $g, h, k \in G$. With respect to the right compatible partial order $\leqslant$ defined by (2.1), we have: for all $g, h, k \in G$, the least upper bound $g \vee h$ exists in G if and only if $(gk) \vee (hk)$ exists in G, in which case $(gk) \vee (hk) = (g \vee h)k$.*

Proof. Let $g, h, k \in G$ and suppose that $g \vee h$ exists in G. We claim that the element $(g \vee h)k$ of G is the least upper bound of gk and hk under $\leqslant$. Since $g, h \leqslant g \vee h$ and $\leqslant$ is right compatible, then $gk, hk \leqslant (g \vee h)k$. Suppose that $l \in G$ is such that $gk, hk \leqslant l$. Again from the right compatibility of $\leqslant$, we have $g, h \leqslant lk^{-1}$, so that $g \vee h \leqslant lk^{-1}$. Thus, $(g \vee h)k \leqslant l$ and, therefore, $(gk) \vee (hk) = (g \vee h)k$. Now suppose that $(gk) \vee (hk)$ exists. By the previous argument, $((gk) \vee (hk))k^{-1}$ also exists, and equals $g \vee h$. □

Observe that, from Lemma 2.1.3 (ii) and the previous lemma, we have that, for all $a, b \in R$ and $k \in G$, $a \vee b$ exists in R and $(ak) \vee (bk) = (a \vee b)k$ exists in G.

Although the generalized Bruck-Reilly extension will be formulated in terms of a cancellative monoid with no non-trivial units which satisfies condition (LI), in our application to tiling semigroups a particular case will arise, namely the positive cone of a lattice ordered group. For that reason, we now consider lattice ordered groups and their connection with groups of left quotients.

A group G is a *partially ordered group* if it is a group equipped with a partial order which is right and left compatible with the group operation. A partially ordered group G is a *lattice ordered group* if the underlying order is a lattice order, that is, the least upper bound $\vee$ and the greatest lower bound $\wedge$ of any pair of elements exist. The following criteria will be of use:

Proposition 2.1.6 ([11], Proposition 3.3)**.** *A partially ordered group is a lattice ordered group if and only if the least upper bound $1 \vee g$ exists for each $g \in G$.*

Lemma 2.1.7. *Let $(G, \leqslant')$ be a lattice ordered group. Then the positive cone G^+ of G with respect to $\leqslant'$ is a cancellative monoid with no non-trivial units satisfying condition (LI). Moreover, the partial order $\leqslant$ given by (2.1) with $R = G^+$ defines a partial order on G that coincides with $\leqslant'$.*

Proof. Since $1 \leqslant' 1$, then $1 \in G^+$; since, for all $g, h \in G^+$ we have $1 \leqslant' g$ and $1 \leqslant' h$, which imply that $1 \leqslant' gh$ since $\leqslant'$ is compatible, then $gh \in G^+$. Thus, G^+ is a submonoid of G and so, in particular, a monoid.

Since G is cancellative, so is its submonoid G^+; since 1 is the only unit of G^+, then G^+ has no non-trivial units.

We claim that G^+ satisfies condition (LI). Let $g, h \in G^+$. Since $\vee'$ is a lattice order, then $g \vee' h$ exists in G. Thus, since $1 \leqslant' g$ and $g \leqslant' g \vee' h$, by the transitivity of $\leqslant'$, we have that $1 \leqslant' g \vee' h$, that is, $g \vee' h \in G^+$. Let us show that $G^+g \cap G^+h = G^+(g \vee' h)$; this will yield the desired conclusion. Suppose that $k \in G^+(g \vee' h)$. Then $k = r(g \vee' h)$, for some $r \in G^+$. Since $g \vee' h \geqslant' g$, then $k = r(g \vee' h) \geqslant' rg$, and, since $r \in G^+$ means that $r \geqslant' 1$, then

$rg \geqslant' g$. Thus, $k \geqslant' g$. But then $kg^{-1} \geqslant' 1$, so that $kg^{-1} \in G^+$. Therefore, $k = kg^{-1}g \in G^+g$. Similarly, $k = rh^{-1}h \in G^+h$. Hence, $k \in G^+g \cap G^+h$, and so $G^+(g \vee' h) \subseteq G^+g \cap G^+h$. Conversely, suppose that $k \in G^+g \cap G^+h$. Then $k = rg$, for some $r \in G^+$, and $k = sh$, for some $s \in G^+$. Since $r \geqslant' 1$ implies that $k = rg \geqslant' g$ and $s \geqslant' 1$ implies that $k = sh \geqslant' h$, then $k \geqslant' g \vee' h$ by definition of least upper bound. So $k(g \vee' h)^{-1} \geqslant' 1$, so that $k(g \vee' h)^{-1} \in G^+$. Thus, $k = k(g \vee' h)^{-1}(g \vee' h) \in G^+(g \vee' h)$, and so $G^+g \cap G^+h \subseteq G^+(g \vee' h)$. Therefore, $G^+g \cap G^+h = G^+(g \vee' h)$. Hence, we have that the set of principal left ideals of G^+ forms a semilattice under intersection.

Since $G^+ = \{g \in G\colon g \geqslant' 1\}$ is a cancellative monoid with no non-trivial units satisfying condition (LI), by Lemma 2.1.2, the binary relation $\leqslant$ given by (2.1) with $R = G^+$ defines a partial order on G. We claim that the orders $\leqslant$ and $\leqslant'$ coincide. Let $g, h \in G$. On the one hand, if $g \leqslant h$, then by definition $h \in G^+g$. Thus, $h = rg$, for some $r \in G^+$, so that $g \leqslant' rg = h$, as required. On the other hand, if $g \leqslant' h$, then $1 \leqslant' hg^{-1}$, and so $h = hg^{-1}g \in G^+g$, that is, $g \leqslant h$. □

In fact,

Lemma 2.1.8. *Let R be a cancellative monoid with no non-trivial units satisfying condition (LI) and G a group of left quotients of R. Then, the partial order $\leqslant$ defined by (2.1) is compatible with multiplication on the left if and only if $Rg = gR$, for all $g \in G$. In that case, G is a lattice ordered group with*

$$(a^{-1}b) \vee (c^{-1}d) = (a \vee c)^{-1}[(c * a)b \vee (a * c)d]\,,$$

for every $a, b, c, d \in R$.

Proof. Suppose $Rg = gR$, for all $g \in G$. Let $g, h, k \in G$ be such that $g \leqslant h$. Then $h \in Rg$, and so $kh \in k(Rg) = (kR)g = (Rk)g = Rkg$, by assumption. Therefore, $kg \leqslant kh$ and so $\leqslant$ is left compatible. Conversely, suppose $\leqslant$ is compatible with multiplication on the left. Let $g \in G$. We claim that $Rg = gR$. Let $ag \in Rg$, with $a \in R$. Then $g \leqslant ag$, by definition of $\leqslant$, and so $1 \leqslant g^{-1}ag$ by the left compatibility of $\leqslant$. Thus, by Lemma 2.1.3 (i), $b = g^{-1}ag \in R$, which implies that $ag = gb \in gR$. Therefore, $Rg \subseteq gR$. Now let $ga \in gR$, with $a \in R$. Since, again by Lemma 2.1.3 (i), $1 \leqslant a$, we have that $g \leqslant ga$ by the left compatibility of $\leqslant$. Therefore, $ga \in Rg$ by definition of $\leqslant$, and so $gR \subseteq Rg$. Hence, $gR = Rg$.

We now show that, in this case, G is a lattice ordered group. Since the partial order is compatible on both sides, then G is a partially ordered group. Thus, in view of Proposition 2.1.6, we need only show that $1 \vee g$ exists for each $g \in G$. Let $g \in G$. Since G is a group of left quotients of R, there exist $a, b \in R$ such that $g = a^{-1}b$. By Lemma 2.1.3 (ii), $a \vee b$ exists in R. Then $a^{-1}(a \vee b)$ is an element in G and it is easy to check that $a^{-1}(a \vee b) = 1 \vee (a^{-1}b)$, in view of the left compatibility of $\leqslant$. Thus, $1 \vee g$ exists.

In view of the previous considerations, and the left dual of Lemma 2.1.5 (which is satisfied in the presence of left compatibility), we may now easily derive that, for all $a, b, c, d \in R$, we have

$$\begin{aligned}(a^{-1}b) \vee (c^{-1}d) &= (a \vee c)^{-1}(a \vee c)[(a^{-1}b) \vee (c^{-1}d)] \\ &= (a \vee c)^{-1}[((c \vee a)a^{-1}b) \vee ((a \vee c)c^{-1}d)] \\ &= (a \vee c)^{-1}[(c * a)b \vee (a * c)d]\,.\end{aligned}$$

□

2.2 A review of the theory of bisimple inverse semigroups

As a motivation for the generalized Bruck-Reilly extension, which will be introduced in the next section, in this section we recall Clifford's description of bisimple inverse monoids, Reilly's generalization of Clifford's result to bisimple inverse semigroups, Reilly's description of bisimple inverse semigroups whose semilattice of idempotents is isomorphic to the non-negative integers under reverse order, and Munn's generalization of this result. In Section 2.3 we will also investigate the connection between these semigroups and generalized Bruck-Reilly extensions.

Recall that a semigroup S is said to be *bisimple* if it has only one $\mathcal{D}$-class, namely S, and that it is said to be *simple* if it has only one $\mathcal{J}$-class; a semigroup with zero is said to be 0-*bisimple* if it has only two $\mathcal{D}$-classes, namely $\{0\}$ and $S \setminus \{0\}$, and 0-*simple* if it has only two $\mathcal{J}$-classes. Of course, as $\mathcal{D} \subseteq \mathcal{J}$, a bisimple semigroup is always simple and a 0-bisimple semigroup is always 0-simple.

The first structural description of bisimple inverse monoids is due to Clifford [8].

Let S be a monoid. Recall that the set of right units of S is a subsemigroup of S, called the *right unit subsemigroup* of S, and the set of all units of S is a subgroup of S, called the *group of units* of S. Clifford showed that the structure of a bisimple inverse monoid is determined by that of its right unit subsemigroup. We now describe this result.

Let R be a right cancellative monoid satisfying condition (LI), that is, the intersection of any two principal left ideals of R is a principal left ideal of R.

Let $a, b \in R$. Notice that, if there is no assumption on the non-trivial units of R, then there may exist distinct elements c and d in R such that $Ra \cap Rb = Rc$ and $Ra \cap Rb = Rd$. Clearly, such elements belong to the same $\mathcal{L}$-class. Fix a transversal T of the set of $\mathcal{L}$-classes. For each a and b in R, denote by $a \vee b$ the member of T in the $\mathcal{L}$-class of an element c satisfying $Ra \cap Rb = Rc$; thus, $a \vee b$ is the unique element in T such that $Ra \cap Rb = R(a \vee b)$. Since $a \vee b$ belongs to Rb, there exists $a * b$ in R, unique since R is right cancellative, such that $a \vee b = (a * b)b$. On the set $R \times R$, define the equivalence relation

$$(a, b) \sim (a', b') \Leftrightarrow \text{ there exists a unit } u \text{ in } R \text{ such that } a' = ua \text{ and } b' = ub\,,$$

and denote by $[a, b]$ the $\sim$-class of (a, b). Finally, on the quotient $(R \times R)/\sim$, denoted by

$R^{-1} \circ R$, consider the operation

$$[a,b][c,d] = [(c*b)a, (b*c)d]\,,$$

for all $a,b,c,d \in R$. Clifford's main result may be stated as follows:

Theorem 2.2.1 ([8], Main Theorem)**.** *Let R be a right cancellative monoid satisfying condition (LI). Then $R^{-1} \circ R$ is a bisimple inverse monoid whose right unit subsemigroup is isomorphic to R. Conversely, if S is a bisimple inverse monoid, then its right unit subsemigroup R is a right cancellative monoid satisfying condition (LI) and $S \simeq R^{-1} \circ R$.*

Remark 2.2.2. Note that, from the definition of $*$ and the fact that R is right cancellative, we have $a*a = 1$, for each $a \in R$. In the process of proving Theorem 2.2.1, Clifford shows that $1*a$, with $a \in R$, is a unit in R ([8, Lemma 4.3]) and that an element of $R^{-1} \circ R$ is an idempotent if and only if it is of the form $[a,a]$ ([8, Lemma 4.11]). Finally, it is easy to check that $[a,b]^{-1} = [b,a]$.

In view of these observations, it is trivial to show that, for all $[a,b],[c,d] \in R^{-1} \circ R$, we have $[a,b]\mathcal{R}[c,d]$ if and only if $a = uc$ for some unit u of R (and, dually, $[a,b]\mathcal{L}[c,d]$ if and only if $b = ud$ for some for some unit u of R).

When R is cancellative, then $R^{-1} \circ R$ is E-unitary and, in fact, the converse also holds. Recall that a necessary and sufficient condition for an inverse semigroup S to be E-unitary is that $\sigma \cap \mathcal{R}$ is the identity relation on S.

Proposition 2.2.3. *Let R be a right cancellative monoid satisfying condition (LI). Then $R^{-1} \circ R$ is an E-unitary inverse semigroup if and only if R is cancellative.*

Proof. Suppose that R is cancellative. By Theorem 2.2.1, $R^{-1} \circ R$ is a (bisimple) inverse monoid; we claim that it is E-unitary. Let $[a,b] \in R^{-1} \circ R$ and $[c,c] \in E_{R^{-1}\circ R}$ be such that $[a,b][c,c] \in E_{R^{-1}\circ R}$. Then

$$[a,b][c,c] = [(c*b)a, (b*c)c] = [(c*b)a, b \vee c] = [(c*b)a, c \vee b] = [(c*b)a, (c*b)b]$$

is an idempotent, which implies that $(c*b)a = (c*b)b$. But then, by assumption, $a = b$ and so $[a,b] = [a,a]$ is an idempotent in $R^{-1} \circ R$. Therefore, $R^{-1} \circ R$ is E-unitary.

Conversely, suppose $R^{-1} \circ R$ is E-unitary. Then R embedds in its maximum group image $(R^{-1} \circ R)/\sigma$. In fact, let $\phi\colon R \to (R^{-1} \circ R)/\sigma$ be the map defined by $a\phi = [1,a]\sigma^\natural$. Notice that, for all $a,b \in R$,

$$[1,a][1,b] = [(1*a)1, (a*1)b] = [1*a, (a*1)1b] = [1*a, (a\vee 1)b] = [1*a, (1\vee a)b] = [1*a, (1*a)ab]\,.$$

Since $1*a$ is a unit in R (cf. Remark 2.2.2), then $[1,a][1,b] = [1,ab]$ by definition of $\sim$. Thus, for all $a,b \in R$,

$$a\phi\, b\phi = [1,a]\sigma^\natural[1,b]\sigma^\natural = ([1,a][1,b])\sigma^\natural = [1,ab]\sigma^\natural = (ab)\phi\,,$$

and so ϕ is a homomorphism. Now $a\phi = b\phi$ if and only if $[1,a]\sigma^\natural = [1,b]\sigma^\natural$, that is, $[1,a]\sigma[1,b]$. But, by Remark 2.2.2, $[1,a]\mathcal{R}[1,b]$, hence the fact that $R^{-1} \circ R$ is E-unitary implies that $[1,a] = [1,b]$. But then $a = b$ by definition of $\sim$, and so ϕ is an embedding. It follows that, if $a, b, c \in R$ are such that $ac = bc$ or $ca = cb$, then, identifying the elements of R with its image under ϕ, we have that $a = b$ in the group $(R^{-1} \circ R)/\sigma$ and hence in R. We thus conclude that R is cancellative. □

It is motivating for the sequel to recall that Clifford notes that if R is the positive cone of a lattice ordered group, then R is in fact a (right and left) cancellative monoid in which both the set of principal left ideals and the set of principal right ideals form semilattices under intersection.

In [54], Reilly extended Clifford's result by showing that the structure of any bisimple inverse semigroup with or without identity is determined by any of its $\mathcal{R}$-classes.

Let R be a *right partial semigroup*, that is, a set equipped with a partial binary operation satisfying the condition

(RPS) for all $a, b, c \in R$, if the product $a(bc)$ is defined then so is the product $(ab)c$, and in that case $a(bc) = (ab)c$.

We say that $P \subseteq R$ is a *subsemigroup of the right partial semigroup* R if, whenever $p, q \in P$ are such that pq is defined in R, then $pq \in P$. Let P be a subsemigroup of R such that

(P1) for all $a, b \in R$, ab is defined if and only if $a \in P$;

(P2) there exists an element $1 \in P$ which is a left identity of R, that is, for all $a \in R$ we have $1a = a$;

(P3) for all $p, q \in P$ and all $a \in R$, $pa = qa$ implies that $p = q$;

(P4) for all $a, b \in R$ there exists $c \in R$ such that $Pa \cap Pb = Pc$.

Such a pair (R, P) is called an *RP-system*.

For example, if R is a cancellative monoid which satisfies condition (LI), then (R, R) is an RP-system.

Fix an RP-system (R, P). It is clear that the binary relation on $R \times R$ defined by, for $a, b \in R$,

$$(a, b) \in \mathcal{L}' \Leftrightarrow Pa = Pb$$

is an equivalence relation. Fix a transversal V of the set of $\mathcal{L}'$-classes in R. For each pair of elements a and b in R, denote by $a \vee b$ the element from V in the $\mathcal{L}'$-class of an element c satisfying $Pa \cap Pb = Pc$. Further, for $a, b \in R$, denote by $a * b$ the unique element in P such that $(a * b)b = a \vee b$. Finally, identify the elements of $R \times R$ by the rule: for all $(a, b), (a', b') \in R \times R$,

$$(a, b) \sim (a', b') \Leftrightarrow \text{ there exists a unit } u \text{ in } P \text{ such that } a' = ua \text{ and } b' = ub\,.$$

Denote by $[a,b]$ the $\sim$-class of (a,b) and by $R^{-1}\circ R$ the quotient $(R\times R)/\sim$.

Theorem 2.2.4 ([54], Theorem 2.2)**.** *The semigroup $R^{-1}\circ R$ equipped with the operation defined by*

$$[a,b][c,d]=[(c*b)a,(b*c)d]$$

for all $[a,b],[c,d]\in(R\times R)/\sim$, is a bisimple inverse semigroup such that

(i) $E_{R^{-1}\circ R}$ isomorphic to the semilattice of $\mathcal{L}'$-classes of R, under the order $L'_a\leqslant L'_b$ if and only if $Pa\subseteq Pb$, for all $a,b\in R$;

(ii) there exists an $\mathcal{R}$-class R' of $R^{-1}\circ R$ isomorphic to R;

(iii) $R^{-1}\circ R$ has an identity if and only if there exists $a\in R$ such that $Pa=R$, namely $[a,a]$.

Conversely, given any bisimple inverse semigroup S and any idempotent e in S, we have $S\simeq R_e^{-1}\circ R_e$, with $R_e^{-1}\circ R_e$ constructed as described above from the RP-system (R_e,P_e), where R_e is the $\mathcal{R}$-class of e, which is a right partial semigroup with respect to the operation

$$\forall a,b\in R_e,\quad a\cdot b \text{ is defined } \Leftrightarrow ab\in R_e, \text{ in which case } a\cdot b=ab\,,$$

and P_e is the right unit subsemigroup of R_e, that is, the set of all elements $u\in eSe$ such that $uv=e$ for some v.

Remark 2.2.5. Not surprisingly, in view of Clifford's findings (cf. Remark 2.2.2), Reilly shows that an element of $R^{-1}\circ R$ is an idempotent if and only if it is of the form $[a,a]$ and that $[a,b]^{-1}=[b,a]$.

Also,

Proposition 2.2.6 ([53], Lemma 1.4)**.** *Let (R,P) be an RP-system. Then, in $R^{-1}\circ R$,*

(i) $[a,b]\mathcal{R}[c,d]$ if and only if $a=uc$ for some unit u of P, or, equivalently, if and only if $Pa=Pc$;

(ii) $[a,b]\mathcal{L}[c,d]$ if and only if $b=vd$ for some unit v of P, or, equivalently, if and only if $Pb=Pd$;

(iii) $[a,b]\mathcal{H}[c,d]$ if and only if $a=uc$ and $b=vd$ for some units u,v of P, or, equivalently, if and only if $Pa=Pc$ and $Pb=Pd$.

Note that if R has no non-trivial units, then the relation $\sim$ is the identity on $R\times R$ and so $R^{-1}\circ R\simeq R\times R$ under the multiplication

$$(a,b)(c,d)=((c*b)a,(b*c)d)\,.$$

Reilly notes that any pair (R, P), with R a lattice ordered group and P its positive cone, is an RP-system. Also, if (R, P) is an RP-system with R a group, then P is cancellative and R is its group of left quotients.

An inverse semigroup S is an *ω-semigroup* if its semilattice of idempotents E_S is isomorphic to the non-negative integers under the reverse order, that is, $E_S \simeq C_\omega = \{e_0 > e_1 > e_2 > \ldots\}$. In the particular case of a bisimple inverse ω-semigroup, Reilly proved the following:

Theorem 2.2.7 ([52], Theorem 3.5)**.** *Let H be a group and $\phi\colon H \to H$ an endomorphism. Then the set $\mathbb{Z}_0^+ \times H \times \mathbb{Z}_0^+$, equipped with the operation*

$$(m, u, n)(p, v, q) = (m - n + t, u\phi^{t-n}\, v\phi^{t-p}, q - p + t)$$

where $t = max(n, p)$ and ϕ^0 is the identity endomorphism of H, is a bisimple inverse ω-semigroup, denoted $B(H, \phi)$. Conversely, every bisimple inverse ω-semigroup is isomorphic to a semigroup constructed in this manner, for some group H and some endomorphism ϕ of H.

The semigroup $B(H, \phi)$ is called the *Bruck-Reilly extension of the group H determined by the endomorphism ϕ of H* because of Reilly's Theorem 2.2.7 and the fact that the first appearance of a construction of this type is due to Bruck [5] for the particular case when ϕ maps every element to the identity.

The bicyclic monoid is an easy example of such a semigroup; it can be obtained by simply taking H to be the trivial group.

A very important generalization of Theorem 2.2.7 is due to Munn.

Theorem 2.2.8 ([46], Theorem 3.1)**.** *Let T be a monoid and $\phi\colon T \to H_1$ a homomorphism from T into its group of units. Then the set $\mathbb{Z}_0^+ \times T \times \mathbb{Z}_0^+$, equipped with the operation*

$$(m, u, n)(p, v, q) = (m - n + t, u\phi^{t-n}\, v\phi^{t-p}, q - p + t)$$

where $t = max(n, p)$ and ϕ^0 is the map defined by $u\phi = 1$, for all $u \in T$, is a simple monoid, denoted by $B(T, \phi)$. Moreover, $B(T, \phi)$ is inverse if and only if T is inverse.

The semigroup $B(T, \phi)$ is called the *Bruck-Reilly extension of the monoid T determined by the homomorphism $\phi\colon T \to H_1$*. As an application of this result, a particular case which was described independently by Kochin [29] and Munn [45] can be deduced: the simple inverse ω-monoids with a predetermined number of $\mathcal{D}$-classes are characterized as the Bruck-Reilly extensions of unions of finite chains of groups, the length of the chain being equal to the number of $\mathcal{D}$-classes (see, for example, [46, Theorem 3.6]).

2.3 The construction

We shall describe a new extension of a semigroup T, starting with a cancellative monoid R with no non-trivial units which satisfies condition (LI) (cf. Section 2.1) and which *acts* *reversely on T by homomorphisms*, in the sense that there exists a mapping $\theta\colon R \to \operatorname{End} T$ such that, for all $a, b \in R$ and $u, v \in T$, we have

(AH1) $(uv)\theta_a = (u\theta_a)(v\theta_a)$;

(AH2) $u\theta_{ab} = (u\theta_b)\theta_a$;

(AH3) $u\theta_1 = u$ (that is, θ_1 is the identity endomorphism of T),

where, for clarity in the sequel, we are denoting by θ_a the endomorphism $a\theta$. A mapping θ satisfying the above conditions is called an *anti-homomorphism* of R into the monoid of endomorphisms of T. The following is the main result of this section.

Theorem 2.3.1. *Let R be a cancellative monoid with no non-trivial units whose principal left ideals form a semilattice under intersection, $(T, \cdot)$ a semigroup and $\theta\colon R \to \operatorname{End} T$ an anti-homomorphism. Then $S(R, T, \theta) = R \times T \times R$ is a semigroup with respect to the operation defined by: for all $(a, u, b), (c, v, d) \in R \times T \times R$,*

$$(a, u, b)(c, v, d) = ((c * b)a, u\theta_{c*b} \cdot v\theta_{b*c}, (b * c)d)\,.$$

Proof. Recall from Section 2.1 that, since R is a cancellative monoid with no non-trivial units which satisfies condition (LI), then we can consider a group G of left quotients of R and, on G, a right compatible partial order defined by $g \leqslant h$ if and only if $h \in Rg$, for all $g, h \in G$. Moreover, by Lemma 2.1.3 (ii) for all $a, b \in R$, the least upper bound $a \vee b$ of a and b under $\leqslant$ belongs to R and, since $Ra \cap Rb = R(a \vee b)$, there exist elements $a * b, b * a$ in R with the property that $a \vee b = (a * b)b = (b * a)a$.

In order to prove that the operation is associative, let $(a, u, b), (c, v, d)$ and (e, w, f) be elements from $S(R, T, \theta)$. We claim that

$$\big((a, u, b)(c, v, d)\big)(e, w, f) = (a, u, b)\big((c, v, d)(e, w, f)\big)\,.$$

Since, by assumption, R is, in particular, a right cancellative monoid satisfying condition (LI), from Theorem 2.2.1 we may consider the (bisimple inverse) monoid $R^{-1} \circ R$. And, since the first and third components combine in $S(R, T, \theta)$ as in $R^{-1} \circ R$, thus assuring associativity, we need only consider the middle coordinate.

On the one hand, we have that

$$\begin{aligned}\big((a, u, b)(c, v, d)\big)(e, w, f) &= ((c * b)a, u\theta_{c*b} \cdot v\theta_{b*c}, (b * c)d)(e, w, f)\\ &= (\cdots, (u\theta_{c*b} \cdot v\theta_{b*c})\theta_{e*[(b*c)d]} \cdot w\theta_{[(b*c)d]*e}, \cdots)\,,\end{aligned}$$

and so the middle coordinate of $\big((a, u, b)(c, v, d)\big)(e, w, f)$ is

$$\begin{aligned}&(u\theta_{c*b} \cdot v\theta_{b*c})\theta_{e*[(b*c)d]} \cdot w\theta_{[(b*c)d]*e} =\\ &= (u\theta_{c*b})\theta_{e*[(b*c)d]} \cdot (v\theta_{b*c})\theta_{e*[(b*c)d]} \cdot w\theta_{[(b*c)d]*e} && \text{(by (AH1))}\\ &= u\theta_{(e*[(b*c)d])(c*b)} \cdot v\theta_{(e*[(b*c)d])(b*c)} \cdot w\theta_{[(b*c)d]*e} && \text{(by (AH2))}\,;\end{aligned}$$

on the other hand, we have

$$\begin{aligned}(a,u,b)\big((c,v,d)(e,w,f)\big) &= (a,u,b)((e*d)c, v\theta_{e*d}\cdot w\theta_{d*e}, (d*e)f)\\ &= (\cdots, u\theta_{[(e*d)c]*b}\cdot(v\theta_{e*d}\cdot w\theta_{d*e})\theta_{b*[(e*d)c]}, \cdots),\end{aligned}$$

and so the middle coordinate of $(a,u,b)\big((c,v,d)(e,w,f)\big)$ is

$$\begin{aligned}&u\theta_{[(e*d)c]*b}\cdot(v\theta_{e*d}\cdot w\theta_{d*e})\theta_{b*[(e*d)c]} =\\ &= u\theta_{[(e*d)c]*b}\cdot(v\theta_{e*d})\theta_{b*[(e*d)c]}\cdot(w\theta_{d*e})\theta_{b*[(e*d)c]} && \text{(by (AH1))}\\ &= u\theta_{[(e*d)c]*b}\cdot v\theta_{(b*[(e*d)c])(e*d)}\cdot w\theta_{(b*[(e*d)c])(d*e)} && \text{(by (AH2))}.\end{aligned}$$

Thus, it suffices to show that

$$\begin{cases}(e*[(b*c)d])(c*b) = [(e*d)c]*b\\ (e*[(b*c)d])(b*c) = (b*[(e*d)c])(e*d)\\ [(b*c)d]*e = (b*[(e*d)c])(d*e).\end{cases} \tag{2.2}$$

In G, the group of left quotients of R, we have, first,

$$\begin{aligned}(e*[(b*c)d])(c*b) &= (e\vee[(b*c)d])[(b*c)d]^{-1}(c\vee b)b^{-1}\\ &= (e\vee[(b\vee c)c^{-1}d])d^{-1}[(b\vee c)c^{-1}]^{-1}(c\vee b)b^{-1}\\ &= (e\vee(bc^{-1}d)\vee d)d^{-1}c(b\vee c)^{-1}(c\vee b)b^{-1}\\ &= [(ed^{-1}c)\vee b\vee c]b^{-1},\end{aligned}$$

while

$$\begin{aligned}[(e*d)c]*b &= ([(e*d)c]\vee b)b^{-1}\\ &= ([(e\vee d)d^{-1}c]\vee b)b^{-1}\\ &= [(ed^{-1}c)\vee c\vee b]b^{-1};\end{aligned}$$

secondly,

$$\begin{aligned}(e*[(b*c)d])(b*c) &= (e\vee[(b*c)d])[(b*c)d]^{-1}(b*c)\\ &= (e\vee[(b\vee c)c^{-1}d])d^{-1}(b*c)^{-1}(b*c)\\ &= [e\vee(bc^{-1}d)\vee d]d^{-1}\\ &= (ed^{-1})\vee(bc^{-1})\vee 1,\end{aligned}$$

while

$$\begin{aligned}(b*[(e*d)c])(e*d) &= (b\vee[(e*d)c])[(e*d)c]^{-1}(e*d)\\ &= (b\vee[(e\vee d)d^{-1}c])c^{-1}(e*d)^{-1}(e*d)\\ &= [b\vee(ed^{-1}c)\vee c]c^{-1}\\ &= (bc^{-1})\vee(ed^{-1})\vee 1;\end{aligned}$$

finally,

$$\begin{aligned}[(b*c)d]*e &= ([(b*c)d]\vee e)e^{-1}\\ &= ([(b\vee c)c^{-1}d]\vee e)e^{-1}\\ &= [(bc^{-1}d)\vee d\vee e]e^{-1}\,,\end{aligned}$$

while

$$\begin{aligned}(b*[(e*d)c])(d*e) &= (b\vee[(e*d)c])[(e*d)c]^{-1}(d\vee e)e^{-1}\\ &= (b\vee[(e\vee d)d^{-1}c])c^{-1}[(e\vee d)d^{-1}]^{-1}(d\vee e)e^{-1}\\ &= [b\vee(ed^{-1}c)\vee c]c^{-1}d(e\vee d)^{-1}(d\vee e)e^{-1}\\ &= [(bc^{-1}d)\vee e\vee d]e^{-1}\,.\end{aligned}$$

Therefore, (2.2) holds in G. Since condition (LI) implies right reversibility, we have that R embeds in G and so (2.2) holds in R as well. Hence, the operation is associative. $\square$

Definition 2.3.2. Given a cancellative monoid R with no non-trivial units whose principal left ideals form a semilattice under intersection, a semigroup $(T,\cdot)$, and an anti-homomorphism $\theta\colon R\to \operatorname{End}T$, we call $S(R,T,\theta)$ the *generalized Bruck-Reilly extension of the semigroup T by the anti-homomorphism θ.*

In fact, $S(R,T,\theta)$ extends T in the sense that T can be embedded into $S(R,T,\theta)$:

Proposition 2.3.3. *The semigroup $S(R,T,\theta)$ contains a subsemigroup isomorphic to T.*

Proof. Consider the subset $T'=\{(1,u,1)\colon u\in T\}$ of $S(R,T,\theta)$. Since, by (AH3),

$$(1,u,1)(1,v,1)=((1*1)1,u\theta_{1*1}\cdot v\theta_{1*1},(1*1)1)=(1,u\cdot v,1)\,,$$

we conclude that T' is a subsemigroup of $S(R,T,\theta)$ and that the (obviously bijective) mapping from T' to T that maps $(1,u,1)$ to u is a homomorphism. Therefore, $T'\simeq T$. $\square$

Also, the class of generalized Bruck-Reilly extensions contains the class of Bruck-Reilly extensions:

Example 2.3.4. First, note that if T is a semigroup and ϕ is an endomorphism of T, then, for each non-negative integer n, the map $\phi^n\colon T\to T$ is an endomorphism (where, as usual, ϕ^0 is the identity endomorphism of T). Thus, the map $\theta\colon \mathbb{Z}_0^+\to\operatorname{End}T$ defined by $\theta_n=n\theta=\phi^n$, for all $n\in\mathbb{Z}_0^+$, is an anti-homomorphism. In fact, condition (AH1) simply states that θ_n is a homomorphism, for all $n\in\mathbb{Z}_0^+$; condition (AH2) is readily verified: for all $m,n\in\mathbb{Z}_0^+$ and all $u\in T$, we have

$$(t\theta_m)\theta_n=(t\phi^m)\phi^n=t\phi^{m+n}=t\phi^{n+m}=t\theta_{n+m}\,;$$

condition (AH3) holds by convention. (Since $\mathbb{Z}_0^+$ is commutative, θ is in fact a homomorphism.)

This observation will allow us to show that every Bruck-Reilly extension is a generalized Bruck-Reilly extension. Let H be a group, $\phi\colon H \to H$ an endomorphism of H, and consider the Bruck-Reilly extension $B(H,\phi)$ of H determined by ϕ. We claim that $B(H,\phi)$ coincides with $S(\mathbb{Z}_0^+, H, \theta)$, where $\theta\colon \mathbb{Z}_0^+ \to \operatorname{End} H$ is defined as in the preceding paragraph. Notice that $R = \mathbb{Z}_0^+$ is indeed a cancellative monoid, with a single (trivial) unit, namely 0, and where, for all $m, n \in \mathbb{Z}_0^+$, we have

$$(\mathbb{Z}_0^+ + m) \cap (\mathbb{Z}_0^+ + n) = \mathbb{Z}_0^+ + max(m,n)\,.$$

This implies that $\mathbb{Z}_0^+$ satisfies condition (LI) and that, in the notation introduced in Section 2.1, $m \vee n = max(m,n)$ and $m * n = max(m,n) - n$, for all $m, n \in \mathbb{Z}_0^+$. Thus, we can construct the semigroup $S(\mathbb{Z}_0^+, H, \theta)$.

It is straightforward to check that $S(\mathbb{Z}_0^+, H, \theta)$ is exactly the Bruck-Reilly extension $B(H,\phi)$. In fact, both semigroups have base set $\mathbb{Z}_0^+ \times H \times \mathbb{Z}_0^+$ and the operation yields, in $S(R,T,\theta)$,

$$\begin{aligned}(m,u,n)(p,v,q) &= ((p*n)m, u\theta_{p*n}\cdot v\theta_{n*p}, (n*p)q)\\ &= (max(p,n) - n + m, u\theta_{max(p,n)-n}\cdot v\theta_{max(n,p)-p}, max(n,p) - p + q)\\ &= (m - n + t, u\phi^{t-n}\cdot v\phi^{t-p}, q - p + t)\,,\end{aligned}$$

where $t = max(n,p) = max(p,n)$, as in $B(H,\phi)$.

In fact, each semigroup $S(R,T,\theta)$ with $R = \mathbb{Z}_0^+$ and T a group is a Bruck-Reilly extension. By condition (AH1), the map $\phi = \theta_1$ is an endomorphism of the group T, and so we can consider the Bruck-Reilly extension $B(T,\phi)$. To show that both semigroups coincide, it suffices to show that $\phi^n = \theta_n$, for all $n \in \mathbb{Z}_0^+$. In fact, for all $u \in T$, we have

$$u\phi^n = ((u\underbrace{\phi)\phi\ldots)\phi}_{n} = ((u\underbrace{\theta_1)\theta_1\ldots)\theta_1}_{n} = u\theta_{\underbrace{1+1+\ldots+1}_{n}} = u\theta_n\,,$$

by condition (AH2). We thus have our conclusion.

In particular, taking H to be the trivial group and $\theta\colon \mathbb{Z}_0^+ \to \operatorname{End} T$ the trivial anti-homomorphism (defined from the trivial endomorphism $\phi\colon H \to H$ as in the previous example), the extension $S(\mathbb{Z}_0^+, H, \theta)$ yields the bicyclic monoid.

By Example 2.3.4 and Theorem 2.2.7, the class of generalized Bruck-Reilly extensions contains the class of bisimple inverse ω-semigroups. Moreover, in view of Theorem 2.2.8, we also have that the class of generalized Bruck-Reilly extensions contains all simple monoids and, in particular, all simple inverse ω-monoids with a given number of $\mathcal{D}$-classes:

Example 2.3.5. Along the same lines as in Example 2.3.4, given a monoid T, its group of units H_1, and a homomorphism $\phi\colon T \to H_1$, the mapping $\theta\colon \mathbb{Z}_0^+ \to \operatorname{End} H_1$ defined by $n\theta = \phi^n$, for all $n \in \mathbb{Z}_0^+$, is an anti-homomorphism (in fact, again a homomorphism), and we have that $S(\mathbb{Z}_0^+, T, \theta)$ is the Bruck-Reilly extension $B(T,\phi)$ of the monoid T determined by the homomorphism $\phi\colon T \to H_1$. Therefore, the class of extensions we introduced in

Theorem 2.3.1 includes all simple monoids and, in particular, all simple inverse ω-monoids with a given number of $\mathcal{D}$-classes.

With respect to the class of bisimple inverse semigroups, characterized by Reilly in Theorem 2.2.4, there is also something to be said. Recall that, given a cancellative monoid satisfying condition (LI), the pair (R, R) is an RP-system and, thus, we can consider the bisimple inverse semigroup $R^{-1} \circ R$. Also, if R does not have non-trivial units, then $R^{-1} \circ R = R \times R$ under the operation defined by $(a,b)(c,d) = ((c * b)a, (b * c)d)$, for all $(a,b), (c,d) \in R^{-1} \circ R$.

Proposition 2.3.6. *Let R be a cancellative monoid with no non-trivial units whose principal left ideals form a semilattice under intersection, $(T, \cdot)$ a semigroup, $\theta \colon R \to \operatorname{End} T$ an anti-homomorphism, and $S(R,T,\theta)$ the generalized Bruck-Reilly extension of T by θ. Then $R^{-1} \circ R$ is a homomorphic image of $S(R,T,\theta)$.*

Proof. Consider the map $\varphi \colon S(R,T,\theta) \to R^{-1} \circ R$ defined by $(a,u,b)\varphi = (a,b)$, for all $(a,u,b) \in S(R,T,\theta)$. Since, as we saw in the proof of Theorem 2.3.1, the outer coordinates of the elements in $S(R,T,\theta)$ multiply as in $R^{-1} \circ R$, we conclude that φ is a homomorphism. As φ is clearly surjective, we conclude that it is an epimorphism. $\square$

In passing, we note that the following is a sufficient condition for $S(R,T,\theta)$ to contain a subsemigroup isomorphic to $R^{-1} \circ R$:

Proposition 2.3.7. *Let R be a cancellative monoid with no non-trivial units whose principal left ideals form a semilattice under intersection, $(T, \cdot)$ a semigroup, $\theta \colon R \to \operatorname{End} T$ an anti-homomorphism, and $S(R,T,\theta)$ the generalized Bruck-Reilly extension of T by θ. If T is a monoid and each homomorphism θ_a, with $a \in R$, is a monoid homomorphism, then $V = \{(p,1,q) \colon p, q \in R\}$ is a subsemigroup of $S(R,T,\theta)$ isomorphic to $R^{-1} \circ R$.*

Proof. We claim that the subset V of $S(R,T,\theta)$ is a subsemigroup of $S(R,T,\theta)$ and that the map $\psi = \varphi|_V \colon V \to R^{-1} \circ R$, with φ as in the proof of Proposition 2.3.6, is an isomorphism. In fact, we have that

$$(a,1,b)(c,1,d) = ((c * b)a, 1\theta_{c*b} \cdot 1\theta_{b*c}, (b * c)d) = ((c * b)a, 1, (b * c)d)\,,$$

is an element of V, since, under our assumption, θ_{c*b} and θ_{b*c} are monoid homomorphisms. Thus, V is a subsemigroup of $S(R,T,\theta)$. That ψ is a homomorphism is a consequence of being the restriction of a homomorphism to a subsemigroup. It is straightforward to check that ψ is bijective. We conclude that $V \simeq R^{-1} \circ R$. $\square$

Next, we prove some basic properties of generalized Bruck-Reilly extensions. The fact that the map $\varphi \colon (a,u,b) \mapsto (a,b)$ from $S(R,T,\theta)$ onto $R^{-1} \circ R$ is an epimorphism (cf. Proposition 2.3.6) provides some valuable information. Recall that an element of $R^{-1} \circ R$,

$$\begin{array}{ccc} T \simeq T' & \hookrightarrow & S(R,T,\theta) \\ & & \downarrow \\ & & R^{-1} \circ R \end{array}$$

Figure 2.1: Connection between $S(R,T,\theta)$, T and $R^{-1} \circ R$.

with R a cancellative monoid with no non-trivial units which satisfies condition (LI), is an idempotent if and only if it is of the form (a,a) and that $(a,b)^{-1} = (b,a)$ (cf. Remark 2.2.5).

For a semigroup $(T,\cdot)$, consider the following condition on T:

(MR) $u \in (u \cdot T) \cap (T \cdot u)$, for all $u \in T$.

Note that, if T is a monoid or a regular semigroup, then condition (MR) is satisfied.

Proposition 2.3.8. *Let R be a cancellative monoid with no non-trivial units whose principal left ideals form a semilattice under intersection, $(T,\cdot)$ a semigroup, $\theta\colon R \to \operatorname{End} T$ an anti-homomorphism, and $S = S(R,T,\theta)$ the generalized Bruck-Reilly extension of T by θ.*

(1) The idempotents of S are the elements of the form (a,e,a) with e an idempotent of T and the natural partial order on E_S is given by

$$(a,e,a) \leqslant (b,f,b) \Leftrightarrow b \leqslant a \text{ and } e \leqslant f\theta_{ab^{-1}}\,.$$

(2) The inverses of an element (a,u,b) of S are the elements (b,v,a) with v an inverse of u in T.

(3) Suppose that T satisfies (MR). For $a,b,c,d \in R$ and $u,v \in S$, we have

(a) $(a,u,b)\mathcal{R}_S(c,v,d)$ if and only if $a = c$ and $u\mathcal{R}_T v$;

(b) $(a,u,b)\mathcal{L}_S(c,v,d)$ if and only if $b = d$ and $u\mathcal{L}_T v$;

(c) $(a,u,b)\mathcal{H}_S(c,v,d)$ if and only if $a = c$, $b = d$ and $u\mathcal{H}_T v$;

(d) $(a,u,b)\mathcal{D}_S(c,v,d)$ if and only if $u\mathcal{D}_T v$.

(4) If T satisfies (MR), then for each idempotent (a,e,a) the maximal subgroup $H_{(a,e,a)}$ of S is isomorphic to the maximal subgroup H_e of T.

(5) If T satisfies (MR), then S is bisimple if and only if T is bisimple.

(6) S is regular if and only if T is regular.

(7) S is orthodox if and only if T is orthodox.

(8) S is inverse if and only if T is inverse.

(9) *S is an E-unitary inverse semigroup if and only if T is E-unitary inverse semigroup and θ_a is idempotent-pure for each $a \in R$. In particular, if T is a semilattice and θ_a is idempotent-pure for each $a \in R$, then S is E-unitary.*

Proof. Parts (c) and (d) of (3) are immediate consequence of parts (a) and (b), as $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$ and $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$; (5) follows from part (d) of (3), in view of the definition of bisimple semigroup; (6) and (8) are immediate from (2).

(1) Suppose (a, u, b) is an idempotent in S. Since epimorphisms map idempotents to idempotents, we have that (a, b) is an idempotent in $R^{-1} \circ R$, and so $a = b$. Thus, since $a * a = 1$ by Lemma 2.1.4, we have

$$(a, u, a) = (a, u, a)(a, u, a) = ((a * a)a, u\theta_{a*a} \cdot u\theta_{a*a}, (a * a)a) = (a, u\theta_1 \cdot u\theta_1, a)$$

if and only if $u \cdot u = u\theta_1 \cdot u\theta_1 = u$. Therefore,

$$E_S = \{(a, e, a) \in S \colon e \in E_T\} .$$

Now let $(a, e, a), (b, f, b) \in E_S$. By definition, $(a, e, a) \leqslant (b, f, b)$ if and only if

$$(a, e, a) = (a, e, a)(b, f, b) = (b, f, b)(a, e, a) .$$

Thus, $(a, e, a) \leqslant (b, f, b)$ if and only if

$$\begin{cases} (a, e, a) = ((b * a)a, e\theta_{b*a} \cdot f\theta_{a*b}, (a * b)b) = (b \vee a, e\theta_{b*a} \cdot f\theta_{a*b}, a \vee b) \\ (a, e, a) = ((a * b)b, f\theta_{a*b} \cdot e\theta_{b*a}, (b * a)a) = (a \vee b, f\theta_{a*b} \cdot e\theta_{b*a}, b \vee a) . \end{cases}$$

In this case, $a = b \vee a\, (= a \vee b)$, and so $b \leqslant a$. Now, for $b \leqslant a$, we get $b * a = (b \vee a)a^{-1} = 1$ and $a * b = (a \vee b)b^{-1} = ab^{-1}$ (notice that ab^{-1} is a representation of the element $a * b$ from R in a group of left quotients of R). Therefore, $(a, e, a) \leqslant (b, f, b)$ if and only if $b \leqslant a$ and

$$\begin{cases} e = e\theta_{b*a} \cdot f\theta_{a*b} = e\theta_1 \cdot f\theta_{ab^{-1}} = e \cdot f\theta_{ab^{-1}} \\ e = f\theta_{a*b} \cdot e\theta_{b*a} = f\theta_{ab^{-1}} \cdot e\theta_1 = f\theta_{ab^{-1}} \cdot e . \end{cases} \tag{2.3}$$

Since ab^{-1} is an element of R, f is an idempotent in T, and $\theta_{ab^{-1}} \colon T \to T$ is a homomorphism, then $f\theta_{ab^{-1}} \in E_T$. Hence, $e \leqslant f\theta_{ab^{-1}}$. Conversely, this inequality implies (2.3). We thus have the desired conclusion.

(2) Let $(a, u, b) \in S$, and let $(c, v, d) \in S$ be an inverse of (a, u, b). Then

$$\begin{cases} (a, u, b)(c, v, d)(a, u, b) = (a, u, b) \\ (c, v, d)(a, u, b)(c, v, d) = (c, v, d) . \end{cases}$$

Thus, applying the epimorphism φ from Proposition 2.3.6, we conclude that $(a,b)(c,d)(a,b) = (a,b)$ and that $(c,d)(a,c)(c,d) = (c,d)$ in $R^{-1} \circ R$. Therefore, by Remark 2.2.5, we have that $(c,d) = (b,a)$. Now,

$$\begin{aligned}
(a,u,b) &= (a,u,b)(b,v,a)(a,u,b) \\
&= ((b*b)a, u\theta_{b*b} \cdot v\theta_{b*b}, (b*b)a)(a,u,b) \\
&= (a, u\theta_1 \cdot v\theta_1, a)(a,u,b) \\
&= (a, u \cdot v, a)(a,u,b) \\
&= ((a*a)a, (u \cdot v)\theta_{a*a} \cdot u\theta_{a*a}, (a*a)b) \\
&= (a, u \cdot v \cdot u, b)
\end{aligned}$$

if and only if $u \cdot v \cdot u = u$ and, similarly, $(b,v,a)(a,u,b)(b,v,a) = (b,v,a)$ if and only if $v \cdot u \cdot v = v$. We therefore have our claim.

(3) Assume that T satisfies condition (MR). Let $a,b,c,d \in R$ and $u,v \in T$. We prove only (a), since (b) is analogous (and (c) and (d) are immediate consequences of (a) and (b), as already mentioned).

So suppose $(a,u,b)\mathcal{R}_S(c,v,d)$. If $(a,u,b) = (c,v,d)$, then $a = c$ and $u = v$, which in particular implies that $u\mathcal{R}_T v$, and so we have our claim. Now suppose $(a,u,b) \neq (c,v,d)$. Then, by Proposition 2.2.6 (i), we have $a = c$, as $(a,b)\mathcal{R}_{R^{-1}\circ R}(c,d)$. By definition of $\mathcal{R}$, we have $(a,u,b)x = (a,v,d)$ and $(a,v,d)y = (a,u,b)$, for some $x,y \in S^1$. Since $(a,u,b) \neq (c,v,d)$, we must have $x \neq 1$ and $y \neq 1$, and so $x = (p,w,q)$ and $y = (r,z,s)$, for some $(p,w,q),(r,z,s) \in R \times T \times R$. Thus,

$$\begin{cases} ((p*b)a, u\theta_{p*b} \cdot w\theta_{b*p}, (b*p)q) = (a,v,d) \\ ((r*d)a, v\theta_{r*d} \cdot z\theta_{d*r}, (d*r)s) = (a,u,b)\,. \end{cases}$$

Since R is cancellative, from $(p*b)a = a$, we get that $p*b = 1$ and, from $(r*d)a = a$, we get that $r*d = 1$. Therefore, $u \cdot w\theta_{b*p} = v$ and $v \cdot z\theta_{d*r} = u$, with $w\theta_{b*p}, z\theta_{d*r} \in T$, so that $u\mathcal{R}_T v$.

Conversely, suppose that $a = c$ and $u\mathcal{R}_T v$. If $u \neq v$, then $u \cdot w = v$ and $v \cdot z = u$ for some $w,z \in T$, by definition of $\mathcal{R}$. Then $(b,w,d),(d,z,b) \in S$ are such that

$$(a,u,b)(b,w,d) = ((b*b)a, u\theta_{b*b} \cdot w\theta_{b*b}, (b*b)d) = (a, u \cdot w, d) = (a,v,d)\,,$$

and, similarly, $(a,v,d)(d,z,b) = (a,u,b)$. Therefore, $(a,u,b)\mathcal{R}_S(a,v,d)$, that is, $(a,u,b)\mathcal{R}_S(c,v,d)$. Now suppose $u = v$. Since $u \in u \cdot T$, then $u = u \cdot w$ for some $w \in T$. Thus, $(b,w,d),(d,w,b) \in S$ are such that $(a,u,b)(b,w,d) = (a,u,d)$ and $(a,u,d)(d,w,b) = (a,u,b)$, and so $(a,u,b)\mathcal{R}_S(a,u,d)$. We thus have our conclusion.

(4) By part (c) of (3), we have $H_{(a,e,a)} = \{(a,u,a) \in S \colon u\mathcal{H}_T e\}$. Thus, it is easy to see that the mapping $\beta\colon H_{(a,e,a)} \to H_e$ defined by $(a,u,a)\beta = u$ is a well-defined bijection. It is also straightforward to check that it is a homomorphism: in fact, for all $(a,u,a),(a,v,a) \in H_{(a,e,a)}$, we have

$$((a,u,a)(a,v,a))\beta = ((a*a)a, u\theta_{a*a} \cdot v\theta_{a*a}, (a*a)a)\beta\,,$$

that is,

$$((a,u,a)(a,v,a))\beta = (a, u\cdot v, a)\beta = u\cdot v = (a,u,a)\beta \cdot (a,v,a)\beta\,.$$

Therefore, β is an isomorphism.

(7) Assume that T is orthodox. Then, in particular, T is regular and so, by (6), so is S. Thus, we only have to show that the idempotents of S form a subsemigroup. From (1), let $(a,e,a),(b,f,b) \in E_S$. Then

$$(a,e,a)(b,f,b) = ((b*a)a, e\theta_{b*a} \cdot f\theta_{a*b}, (a*b)b) = (b \vee a, e\theta_{b*a} \cdot f\theta_{a*b}, a \vee b)\,,$$

where $e\theta_{b*a}$ and $f\theta_{a*b}$ are idempotents in T, and so the product $e\theta_{b*a} \cdot f\theta_{a*b} \in E_T$ by assumption. Since $a \vee b = b \vee a$, we have that $(a,e,a)(b,f,b) \in E_S$. Hence, the semigroup S is orthodox. Conversely, assume that S is orthodox. By (6), T is regular. Let $e,f \in E_T$. Then $(1,e,1),(1,f,1) \in E_S$, and so

$$(1,e,1)(1,f,1) = ((1*1)1, e\theta_{1*1} \cdot f\theta_{1*1}, (1*1)1) = (1, e\cdot f, 1)$$

is an idempotent in S. By (2), $e\cdot f$ is an idempotent in T. Therefore, T is also orthodox.

(9) Finally, assume that S is E-unitary; we claim that T is E-unitary and that θ_a is idempotent-pure for each $a \in R$. Let $e \in E_T$ and $u \in T$ be such that $e\cdot u \in E_T$. Then $(1,e,1) \in E_S$ and $(1,u,1) \in S$ are such that

$$(1,e,1)(1,u,1) = ((1*1)1, e\theta_{1*1} \cdot u\theta_{1*1}, (1*1)1) = (1, e\cdot u, 1)$$

is an idempotent in S. Thus, as S is E-unitary, $(1,u,1)$ is also an idempotent in S, and we conclude by (1) that u is an idempotent in T. Therefore, T is E-unitary.

Now let $a \in R$ and $u \in T$. Assume that $u\theta_a$ is idempotent in T. We want to show that $u \in E_T$. Let e be any idempotent in T. We have

$$(1,u,1)(a,e,a) = ((a*1)1, u\theta_{a*1} \cdot e\theta_{1*a}, (1*a)a) = (a, u\theta_a \cdot e\theta_1, a)\,,$$

as $(1*a)a = a$ and $(a*1)1 = a$ (cf. Lemma 2.1.4). Since $u\theta_a$ and e are idempotents, T is a (E-unitary) inverse semigroup and θ_1 is a homomorphism (in fact, the identity homomorphism, by condition (A3)), then $u\theta_a \cdot e\theta_1 = u\theta_a \cdot e$ is idempotent in T. Thus, $(a, u\theta_a \cdot e, a)$ is an idempotent of S and so, as S is E-unitary, $(1,u,1)$ is an idempotent of S as well. By (1), we have that u is idempotent in T, as desired.

Conversely, suppose that T is E-unitary and that θ_a is idempotent-pure for each $a \in R$. Let $(a,e,a) \in E_S$ and $(c,v,d) \in S$ be such that $(a,e,a)(c,v,d) \in E_S$, that is, $((c*a)a, e\theta_{c*a} \cdot v\theta_{a*c}, (a*c)d) \in E_S$. Then $(c*a)a = (a*c)d$ and $e\theta_{c*a} \cdot v\theta_{a*c} \in E_T$. From the first condition we get that $c = d$ in a group G of left quotients of R, and so $c = d$ in R; from the second, we get that $v\theta_{a*c}$ must be an idempotent in T since T is E-unitary. Then, as θ_{a*c} is idempotent-pure, v is an idempotent in T. Therefore, (c,v,d) is an idempotent in S and so S is E-unitary. This concludes the proof.

□

We now investigate when is $S(R,T,\theta)$ a simple semigroup. With R, T, θ and $S = S(R,T,\theta)$ as in Proposition 2.3.8,

Lemma 2.3.9. *If T satisfies condition (MR), then, for all $a,b,c,d \in R$ and $u,v \in T$, we have $(a,u,b) \leqslant_{\mathcal{J}_S} (c,v,d)$ if and only if $u \leqslant_{\mathcal{J}_T} v\theta_r$, for some $r \in R$.*

Proof. Suppose that $(a,u,b) \leqslant_{\mathcal{J}_S} (c,v,d)$. If $(a,u,b) = (c,v,d)$, then, in particular, $u = v$ and, trivially, $u \leqslant_{\mathcal{J}_T} v = v\theta_1$. So assume $(a,u,b) \neq (c,v,d)$. By definition of $\leqslant_{\mathcal{J}}$, we have $(a,u,b) = x(c,v,d)y$ for some $x,y \in S^1$. Assume $y = 1$. Then $x \neq 1$, since $(a,u,b) \neq (c,v,d)$, and so $x = (p,w,q) \in S$. Thus,

$$(a,u,b) = (p,w,q)(c,v,d) = ((c*q)p, w\theta_{c*q} \cdot v\theta_{q*c}, (q*c)d)\,,$$

so that, in particular, $u = w\theta_{c*q} \cdot v\theta_{q*c}$, with $w\theta_{c*q} \in T$, and $r = q*c \in R$. Therefore, $u \leqslant_{\mathcal{L}_T} v\theta_r$ with $r \in R$ and, hence, $u \leqslant_{\mathcal{J}_T} v\theta_r$ with $r \in R$. Similarly, if $x = 1$, then $y \neq 1$ and we conclude that $u \leqslant_{\mathcal{R}_T} v\theta_r$, and so $u \leqslant_{\mathcal{J}_T} v\theta_r$ with $r \in R$. Finally, suppose that $x \neq 1$ and $y \neq 1$. Then $(a,u,b) = (p,w,q)(c,v,d)(r,z,s)$ for some $(p,w,q),(r,z,s) \in S$. Direct calculations (cf., for example, the proof of associativity in Theorem 2.3.1) yield that

$$u = w\theta_{(r*((q*c)d))(c*q)} \cdot v\theta_{(r*((q*c)d))(q*c)} \cdot z\theta_{((q*c)d)r}\,.$$

Since $t = (r*((q*c)d))(q*c) \in R$, we conclude that $u \leqslant_{\mathcal{J}_T} v\theta_t$ with $t \in R$.

Conversely, suppose $u \leqslant_{\mathcal{J}_T} v\theta_r$, for some $r \in R$. Then $u = x \cdot v\theta_r \cdot y$ for some $x,y \in T^1$.

Note that, if $u = w \cdot v\theta_r \cdot z$, for some $w,z \in T$, then $(a,w,rc),(rd,z,b) \in S$ are such that

$$(a,w,rc)(c,v,d)(rd,z,b) = ((c*(rc))a, w\theta_{c*(rc)} \cdot v\theta_{(rc)*c}, ((rc)*c)d)(d,z,b)\,,$$

where

$$c*(rc) = (c \vee (rc))(rc)^{-1} = (cc^{-1}r^{-1}) \vee 1 = r^{-1} \vee 1 = 1$$

and

$$(rc)*c = ((rc) \vee c)c^{-1} = r \vee 1 = r\,.$$

Thus,

$$\begin{aligned}(a,w,rc)(c,v,d)(rd,z,b) &= (a, w\cdot v\theta_r, rd)(rd,z,b)\\ &= (((rd)*(rd))a, (w\cdot v\theta_r)\theta_{(rd)*(rd)}\cdot z\theta_{(rd)*(rd)}, ((rd)*(rd))b)\\ &= (a, w\cdot v\theta_r\cdot z, b)\\ &= (a,u,b)\,,\end{aligned}$$

and, therefore, $(a,u,b) \leqslant_{\mathcal{J}_S} (c,v,d)$. Hence, we have the desired conclusion provided that we show that $u = w\cdot v\theta_r \cdot z$ for some $w,z\in T$.

This is trivially true if $x \neq 1$ and $y\neq 1$; simply take $w = x$ and $y = z$. Now assume that $x\neq 1$ and $y = 1$. Then $u = w\cdot v\theta_r$ for some $w\in T$ (take $w = x$). Since, by assumption, $u \in u\cdot T$, there exists $z\in T$ such that $u = u\cdot z$. But then $u = u\cdot z = (w\cdot v\theta_r)\cdot z = w\cdot v\theta_r\cdot z$, with $w,z\in T$. The case $x = 1$ and $y\neq 1$ is similar. Finally, assume that $x = y = 1$. Then $u = v\theta_r$. As $u\in(u\cdot T)\cap(T\cdot u)$, we have $u = u\cdot z$ and $u = w\cdot u$, for some $w,z\in T$, and, consequently,

$$u = u\cdot z = (w\cdot u)\cdot z = w\cdot u\cdot z = w\cdot v\theta_r\cdot z\,.$$

This completes the proof. □

It follows that

Proposition 2.3.10. *Let R be a cancellative monoid with no non-trivial units whose principal left ideals form a semilattice under intersection, $(T,\cdot)$ a semigroup satisfying condition (MR), $\theta\colon R\to \operatorname{End} T$ an anti-homomorphism, and $S(R,T,\theta)$ the generalized Bruck-Reilly extension of T by θ. Then $S(R,T,\theta)$ is simple if and only if, for all $u,v\in T$, there exists $r\in R$ such that $u\leqslant_{\mathcal{J}_T} v\theta_r$.*

As mentioned before, generalized Bruck-Reilly extensions will be of use for us in connection with tiling semigroups. More precisely, we will give in Section 4.3 a description of a certain kind of tiling semigroup as a Rees quotient of an inverse subsemigroup of a semigroup $S(R,T,\theta)$; the following, straightforward, result will be applied.

Lemma 2.3.11. *Let R be a cancellative monoid with no non-trivial units whose principal left ideals form a semilattice under intersection, $(T,\cdot)$ a semigroup, $\theta\colon R\to\operatorname{End} T$ an anti-homomorphism, and $S(R,T,\theta)$ the generalized Bruck-Reilly extension of T by θ. Let I be a non-empty two-sided ideal of T such that θ_a restricts to an endomorphism of I, for all $a\in R$. Then*

$$J = \{(a,u,b)\in S(R,T,\theta)\colon u\in I\}$$

is a two-sided ideal of $S(R,T,\theta)$.

Proof. By definition, $J\subseteq S(R,T,\theta)$. Since I in non-empty, then so is J. Let $(a,u,b)\in J$ and $(c,v,d)\in S(R,T,\theta)$. Then $(a,u,b)(c,v,d) = ((c*b)a, u\theta_{c*b}\cdot v\theta_{b*c}, (b*c)d)$ is such that

$u\theta_{c*b} \in I\theta_{b*c} \subseteq I$ by assumption, so that $u\theta_{c*b} \cdot v\theta_{b*c} \in IT \subseteq I$ by definition of ideal. Thus, J is a right ideal of $S(R,T,\theta)$. Similarly, $S(R,T,\theta)J \subseteq J$, and we conclude that J is a two-sided ideal of $S(R,T,\theta)$. $\square$

Chapter 3

Tilings and the tiling semigroup

This chapter aims at giving the definitions of all concepts involved in the construction of a tiling semigroup, some background on its origins, a complete review on the research conducted on this subject within the scope of inverse semigroup theory, and also to introduce the class of hypercubic tilings. It is divided into four parts: the first deals with tilings, the second with the tiling semigroup, the third with the research review, and the fourth with the definition of hypercubic tiling.

3.1 Tilings

In this section, we provide all the basic definitions regarding tilings that will play a role in the construction of a tiling semigroup, namely those of tile, tiling, pattern, and finite type tiling. Subsequently, we concentrate on one-dimensional tilings, where the identification between finite type tilings and bi-infinite words over finite alphabets leads to the definition of language of the tiling, a notion that is instrumental in the study of the semigroup.

Let n be a fixed positive integer.

Definition 3.1.1. A *tile* is closed, bounded and connected subset of $\mathbb{R}^n$ which has connected interior, is the closure of its interior, and may carry a decoration. An *(n-dimensional) tiling* $\mathcal{T}$ is a covering of the Euclidean space $\mathbb{R}^n$ by tiles without gaps or overlaps, except, possibly, at the boundary of the tiles.

When thinking of tilings, immediate images and questions come to one's mind. Tilings by Escher (see Figure 3.1, [16]) or the work of Penrose (see Figure 3.2[1]) are unavoidable; questions about regularity or, in particular, periodicity, too.

The idea to retain is that, within the generous boundaries of the definition, many different possibilities are allowed. The following example illustrates, in the particular case of two-dimensional tilings, why all requirements of the definition of a tile translate in fact the natural notion of being the indivisible unit of a tiling.

[1] `http://plus.maths.org/issue45/features/kaplan/Penrose_tiling.gif`

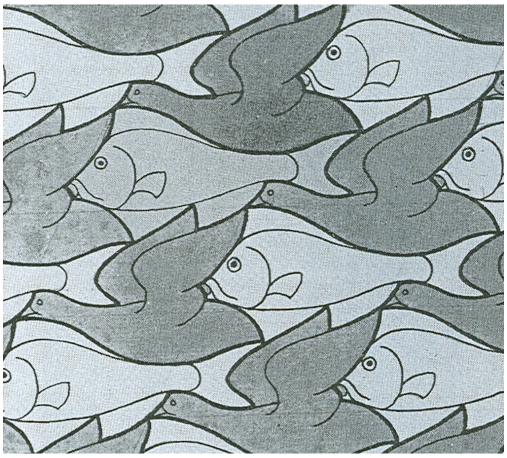


Figure 3.1: Tiling by M. C. Escher

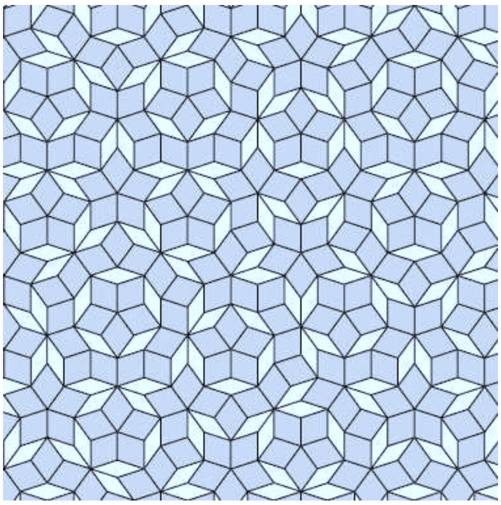
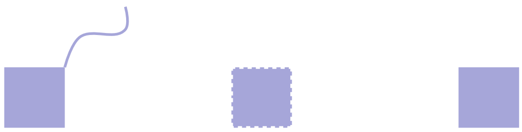


Figure 3.2: Tiling by R. Penrose

Example 3.1.2. In the figure, we have on the left a closed, bounded, and connected subset of $\mathbb{R}^2$, and on the right its interior, which is not a connected set:

Thus, by definition, the object on the left is not a tile, but in fact the union of two tiles. The next figure shows, on the left, a closed, bounded, and connected subset of $\mathbb{R}^2$; at the centre, its interior, which is a connected set; and, on the right, the closure of its interior, which does not coincide with the whole subset:



Again the object on the left is not, by definition, a tile.

Example 3.1.3. The union $\bigcup_{a\in\mathbb{R}} I_a$, where for each $a \in \mathbb{R}$, the set I_a consists of the vertical straight line $\{(x, y) \in \mathbb{R}^2 : x = a\}$, is not a tiling of $\mathbb{R}^2$, as the sets I_a (with $a \in \mathbb{R}$) are not tiles (they are neither closed, nor bounded, nor the closure of their interiors), although $\mathbb{R}^2 = \bigcup_{a\in\mathbb{R}} I_a$.

The leading reference on tilings is the book by Grunbaum and Shephard ([19]).

For the remainder of this section, let $\mathcal{T}$ be a fixed n-dimensional tiling.

Tilings are infinite, but since they are aimed at modelling solids, which are finite entities, and in particular its (finite) parts, we want to consider finite portions of tilings:

Definition 3.1.4. A *pattern in* $\mathcal{T}$ is a finite connected union of tiles from $\mathcal{T}$ with connected interior.

Evidently, the appropriate analogous comments to those made for tiles (cf. Example 3.1.2) could be made for patterns. Note that the way we regard a pattern is in fact exactly as a (connected) set whose elements are tiles: if a is a tile in A, we write $a \in A$. Along the same line, if A is a pattern in $\mathcal{T}$, we write $A \in \mathcal{T}$. Also note that a single tile is, in particular, a pattern.

The obvious operation between patterns is their union. However, this is not the interesting one and in order to achieve a convenient definition we will consider patterns with two (possibly coinciding) distinguished tiles:

Definition 3.1.5. A *doubly pointed pattern* is a pattern together with a fixed ordered pair of tiles belonging to the patter, the first of which is called the *in-tile* and the second the *out-tile*.

If A is a pattern in $\mathcal{T}$ and $a, b \in A$, then the doubly pointed pattern with underlying pattern A, in-tile a, and out-tile b is denoted by (a, A, b).

By definition, two patterns located at different places in the tiling are distinct patterns, even if they "look" the same. The next notion establishes the identification of such patterns:

Definition 3.1.6. A *pattern class* is the class of a pattern under the translation equivalence.

Therefore, two patterns A and B in a tiling $\mathcal{T}$ belong to the same pattern class if there exists a translation by a vector x of $\mathbb{R}^n$ such that $A = B + x$. This is obviously an equivalence relation, and therefore the previous definition is well-posed. We will use the term *tile class* with the obvious meaning: it is the pattern class of a one-tile pattern.

Finally, the corresponding notion for doubly pointed patterns yields what will be the non-zero elements of the tiling semigroup. Thus, we will say that the doubly pointed patterns (a, A, b) and (c, C, d) are *equivalent* if and only if $A = C + x$ and $a = c + x$ and $b = d + x$ for some translation x of $\mathbb{R}^n$. It is trivial to check that this binary relation, defined on the set of all doubly pointed patterns of the tiling, is an equivalence relation. We define:

Definition 3.1.7. A *doubly pointed pattern class* is the equivalence class of a doubly pointed pattern under the equivalence relation above.

If (a, A, b) is a doubly pointed pattern in $\mathcal{T}$, we denote by $[a, A, b]$ its equivalence class. For convenience, we will often refer to A has the *underlying pattern* of $[a, A, b]$ (instead of the accurate, but longer, designation of underlying pattern of the representative (a, A, b)).

Therefore, for any patterns A and C in the same tiling and tiles $a, b \in A$ and $c, d \in C$, we have $[a, A, b] = [c, C, d]$ if and only if $A = C + x$ and $a = c + x$ and $b = d + x$ for some translation x of $\mathbb{R}^n$.

The following is a compactness condition that plays an important role on several occasions, namely in the fact that tiling semigroups will always be finitely generated and in the finiteness of the alphabet of the language associated with a tiling.

Definition 3.1.8. An n-dimensional tiling is said to be of *finite type* if there are only finitely many distinct pattern classes whose underlying pattern has two tiles.

Consequently, in a finite type tiling, there are only finitely many pattern classes whose underlying pattern has only one tile, that is, there are only finitely many tile classes.

Originally, Kellendonk defines a finite type tiling as one in which "the set of connected doubly pointed pattern classes which consist of two tiles is finite" [24]. As we have commented in the Introduction, the fact that Kellendonk requires patterns to be connected has to do with his initial definition of pattern, which did not include that requirement. The important remark to make is that it is equivalent to require that the set of doubly pointed pattern classes which consist of two tiles is finite or that the set of pattern classes which consist of two tiles is finite. In fact, for a fixed tiling $\mathcal{T}$, let X be the set of pattern classes which consist of two tiles and

Y the set of doubly pointed pattern classes which consist of two tiles. By analogy with the notation adopted for doubly pointed patterns classes, denote by $[A]$ the pattern class of a pattern A. Then $X = \{[A_i] \colon i \in I\}$ for some set I, where, to avoid trivialities, we may assume that $[A_i] \neq [A_j]$ for all $i \neq j$. By definition, for each $i \in I$ we have $A_i = \{a_i, b_i\}$ with a_i and b_i distinct tiles in $\mathcal{T}$. But then $Y = \{[a_i, A_i, b_i], [b_i, A_i, a_i], [a_i, A_i, a_i], [b_i, A_i, b_i] \colon i \in I\}$. Since $\#Y = 4\,\#X$, then X and Y are either both finite or infinite, and we have our conclusion.

Example 3.1.9. Suppose a tiling $\mathcal{T}$ in $\mathbb{R}$ contains, for each non-negative integer n, the closed interval of the form $\left[\frac{2^n-1}{2^n}, \frac{2^{n+1}-1}{2^{n+1}}\right]$ — that is, all the intervals $[0, \frac{1}{2}]$, $[\frac{1}{2}, \frac{3}{4}]$, $[\frac{3}{4}, \frac{7}{8}]$, $[\frac{7}{8}, \frac{15}{16}]$, and so on.

Notice that every closed bounded interval of $\mathbb{R}$ fits the definition of tile in $\mathbb{R}$ and that the union $[a, b] \cup [b, c]$ of the tiles $[a, b]$ and $[b, c]$, with $a < b < c$ real numbers, fits the definition of pattern in $\mathbb{R}$.

Thus, each union $P_n = \left[\frac{2^n-1}{2^n}, \frac{2^{n+1}-1}{2^{n+1}}\right] \cup \left[\frac{2^{n+1}-1}{2^{n+1}}, \frac{2^{n+2}-1}{2^{n+2}}\right]$, with $n \geq 0$, is a pattern with two tiles. Moreover, since the length of P_n is

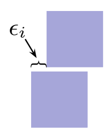
$$L_n = \frac{2^{n+2}-1}{2^{n+2}} - \frac{2^n-1}{2^n} = \frac{2^{n+2}-1-2^2(2^n-1)}{2^{n+2}} = \frac{3}{2^{n+2}}$$

and $L_m = L_n$ if and only if $m = n$, for all non-negative integers m and n, we have that two distinct pairs of consecutive intervals yield distinct pattern classes. Therefore, there exist infinitely many pattern classes in this tiling. Hence, $\mathcal{T}$ is not a finite type tiling.

As Kellendonk notes, (another) equivalent condition for a tiling to be of finite type is that, for every positive real number r, the number of distinct pattern classes of the tiling that fit into a ball of radius r is finite [25]. This condition is easily seen to fail in the tiling in Example 3.1.9, as the ball of centre $\frac{1}{2}$ and radius $\frac{1}{2}$, that is, the open interval $(0, 1)$, contains countably many pattern classes.

Example 3.1.10. Let $(\epsilon_i)_{i \in \mathbb{N}}$ be a strictly increasing sequence of real numbers in the open interval $(0, 1)$. Consider the tiling of $\mathbb{R}^2$ by squares of side 1, horizontally aligned and vertically shifted as suggested in Figure 3.3. In this tiling, there are countably many different doubly pointed pattern classes whose underlying pattern has two tiles, namely

and



for each positive integer i. Equivalently, one could notice that, since $\epsilon_i < 1$ for all $i \in \mathbb{N}$, there are countably many different pattern classes (with two tiles) that the tiling can fit into a ball

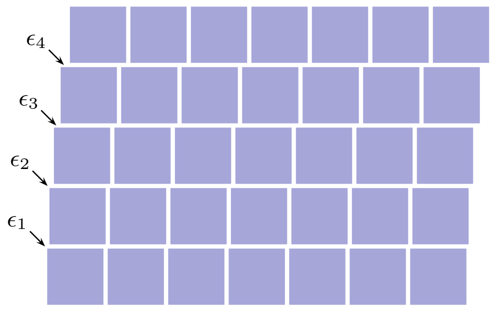
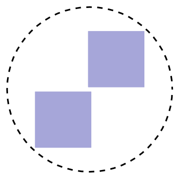


Figure 3.3: A tiling of $\mathbb{R}^2$ which is not of finite type

of radius $\sqrt{2}$:



(although there exists a unique tile class). Therefore, this is an example of a tiling which is not of finite type.

In [25], Kellendonk remarks that finite type tilings have the following property: the set of (doubly pointed) pattern classes of the tiling is countable. In particular, a finite type tiling is a countable union of tiles.

As usual in the literature, we deal only with finite type tilings, even without mentioning it.

The special case of tilings of the real line

We now focus our attention on one-dimensional tilings, or tilings of the real line. Here, an identification between finite type one-dimensional tilings and bi-infinite words over finite alphabets leads to the definition of the language of a tiling, which will prove most important. In addition, we also relate these languages with factorial languages, due to a natural generalization of one-dimensional tiling semigroup that is often used and that will be described in the next section.

In the particular case of one-dimensional tilings, the vast range of possibilities that a tiling can assume is replaced by a rather simpler and neat universe. And if it is true that such tilings find no application in the theory that motivated the birth of the tiling semigroup, we will see that it is nontheless true that considering tiling semigroups of one-dimensional tilings constitutes a great source of inspiration, apart from being an interesting algebraic object on its own. Further, one-dimensional tilings are closely connected with one-dimensional dynamical systems [2]; an evidence of this fact will be produced in Section 6.1.

The reason why one-dimensional tilings are simpler, and thus more tractable, objects has to do with what a tile in $\mathbb{R}$ can be: the only closed, bounded and connected subsets of $\mathbb{R}$ are the closed, bounded intervals of $\mathbb{R}$, which in turn have connected interiors and are the closures of their interiors. Therefore, a (finite type) tiling of the real line is a countable collection of closed, bounded intervals of $\mathbb{R}$ which intersect each other at most on a boundary point and whose union is $\mathbb{R}$.

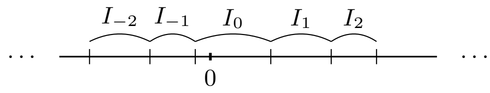


Figure 3.4: One-dimensional tilings and bi-infinite words

The fact that we consider only finite type tilings has, in addition to imposing that the collection of intervals is countable, a striking consequence: the fact that these tilings admit only finitely many tile classes implies that there are only finitely many intervals of different lengths, since two intervals are equivalent under translation if and only if they have the same length. Thus, given a one-dimensional tiling $\mathcal{T} = \bigcup_{i\in\mathbb{Z}}[r_i, s_i]$, the set $\Sigma_{\mathcal{T}} = \{|s_i - r_i| \colon i \in \mathbb{Z}\}$ of lengths of the intervals in the tiling is finite. Therefore, we can associate to $\mathcal{T}$ the bi-infinite word $\alpha_{\mathcal{T}} = \dots a_{-2}a_{-1}a_0a_1a_2\dots$ over the finite alphabet $\Sigma_{\mathcal{T}}$ defined by $a_i = |s_i - r_i|$, for all $i \in \mathbb{Z}$. This identification between one-dimensional tilings and bi-infinite words over finite alphabets is the cornerstone of the study of tiling semigroups of one-dimensional tilings. It is so powerful that we will always regard such tilings as bi-infinite words over finite alphabets.

An instance of the advantage of this point of view is how easy it makes to identify pattern classes in the tiling: since, by definition, a pattern is a finite connected union of tiles, the pattern classes turn out simply to be the finite factors of the bi-infinite word associated with the tiling. The collection of such factors, for a given tiling $\mathcal{T}$, constitutes a language over $\Sigma_{\mathcal{T}}$:

Definition 3.1.11. Let $\mathcal{T}$ be a one-dimensional tiling. The language $L(\mathcal{T})$ of all finite factors of the bi-infinite word $\alpha_{\mathcal{T}}$ associated with the tiling is called the *language of the tiling* $\mathcal{T}$ or the *tiling language of* $\mathcal{T}$.

Example 3.1.12. Consider the one-dimensional tiling $\mathcal{T}$ with the following associated bi-infinite word:

$$\dots a\,b\,a\,b\,a\,b\,a\,b\dots\ ,$$

where a and b are distinct letters. Then $\mathcal{T}$ contains only intervals of two different lengths and each interval has, both to its left and to its right, an interval of the other length. Moreover, the tiling language of $\mathcal{T}$ is the language over $\{a, b\}$ consisting of the words of the form a, b,

$(ab)^n$, $b(ab)^n$, $(ab)^n a$, and $(ba)^n$, where n is a positive integer. That is,

$$L(\mathcal{T}) = (ab)^+ \cup (ba)^+ \cup (ab)^* a \cup b(ab)^* .$$

Note that the elements of $L(\mathcal{T})$ represent the pattern classes of the tiling, as pointed out above, and not its patterns, since knowing that some word if a factor of $\alpha_{\mathcal{T}}$ does not tell us what is its postion in the bi-infinite word. Also note that the class of tiling languages is precisely the class $\mathcal{F}_{biw}$ of languages of all factors of bi-infinite words over some finite alphabet mentioned in Section 1.3. Thus, in particular, the tiling language of a one-dimensional tiling is always a factorial language.

3.2 The tiling semigroup

In this section, we define the tiling semigroup of an n-dimensional tiling and compare the given definition (due to Lawson [32]) with the construction originally proposed by Kellendonk in his earlier papers ([24, 25]). As in the previous section, we then concentrate on one-dimensional tilings. For these, we will be able to adopt a very convenient representation of the elements of the semigroup using the language of the tiling. In addition, we shall describe a generalization of this semigroup for an arbitrary factorial language. In fact, we will work with this more general semigroup for a large part of the investigations into one-dimensional tilings semigroups.

Let n be a fixed positive integer and let $\mathcal{T}$ be a fixed n-dimensional tiling. Consider the following binary relation between the doubly pointed pattern classes of $\mathcal{T}$:

Definition 3.2.1. Let $[a, A, b]$ and $[c, C, d]$ be two doubly pointed pattern classes of $\mathcal{T}$. We say that $[a, A, b]$ and $[c, C, d]$ are *composable* if there exist translations x and y in $\mathbb{R}^n$ such that $(A + x) \cup (C + y)$ is a pattern in $\mathcal{T}$ and $b + x = c + y$.

Notice that the condition "$(A + x) \cup (C + y)$ is a pattern in $\mathcal{T}$" implies, in particular that $A + x$ and $C + y$ are patterns in $\mathcal{T}$ and that they coincide on their overlap. , the condition "$b + x = c + y$" ensures that $(A + x) \cup (C + y)$ is a connected union, since A and B are connected sets by definition of pattern, so that $A+x$ and $B+y$ are connected sets as well, and $b + x = c + y \in (A + x) \cap (C + y)$. Also note that, if $[a, A, b]$ and $[c, C, d]$ are composable and $x, y \in \mathbb{R}^n$ are such that $(A+x)\cup(C+y)$ is a pattern in $\mathcal{T}$, then $(a+x, (A+x)\cup(C+y), d+y)$ is a doubly pointed pattern of $\mathcal{T}$, since $a+x, d+y \in (A+x)\cup(C+y)$. In [32], Lawson shows that its class is independent of the choice of x and y.

This binary relation underlies the partially defined operation on the doubly pointed pattern classes of a tiling, which is the core of the definition of the tiling semigroup of an arbitrary n-dimensional tiling:

Let $S(\mathcal{T})$ be the set consisting of all doubly pointed pattern classes with underlying pattern in $\mathcal{T}$ together with an element 0. On $S(\mathcal{T})$, define the following binary operation:

$$[a, A, b][c, C, d] = \begin{cases} [a + x, (A + x) \cup (C + y), d + y], & \text{if } [a, A, b] \text{ and } [c, C, d] \text{ are composable} \\ & \text{and } x, y \in \mathbb{R}^n \text{ are such that} \\ & (A + x) \cup (C + y) \text{ is a pattern in } \mathcal{T} \\ 0, & \text{otherwise} \end{cases}$$

and all products involving 0 yield 0. Then

Theorem 3.2.2 (Theorem 9.5.1, [32])**.** *The above binary operation is well-defined and endows $S(\mathcal{T})$ with the structure of an inverse semigroup with zero.*

Definition 3.2.3. We call $S(\mathcal{T})$ the *tiling semigroup* of $\mathcal{T}$.

It is important to record that, in the process of proving that a tiling semigroup is an inverse semigroup with zero, Lawson shows that the non-zero idempotents are precisely the doubly pointed pattern classes with coinciding in-tile and out-tile and that the inverse of a non-zero element $[a, A, b]$ is $[b, A, a]$.

As mentioned in the Introduction, in his original construction [24], Kellendonk considers an almost-groupoid instead of an inverse semigroup and does not consider patterns as being necessarily connected sets. As we have already observed in the Introduction, the first aspect does not constitute much of a difference in the present case, for the elements in Kellendonk's almost-groupoid have a unique inverse. Its formulation as an inverse semigroup has a two-fold advantage: on the one hand, it places this object within the scope of inverse semigroup theory; on the other hand, inverse semigroup is the formulation used in the literature on C^*-algebras as operator algebras, see for example Paterson's book [48]. As to the second aspect, although patterns are assumed to be connected in Kellendonk's later joint work with Lawson [26] (as they were in Lawson's account in [32]), no physical consideration is ever made.

Tiling semigroups of tilings of the real line

Not surprisingly (cf. the subsection of Section 3.1 concerning one-dimensional tilings), tiling semigroups of one-dimensional tilings are of special importance within the family of tiling semigroups, since they are easier to study and often serve as source of insight for the general case.

Let $\mathcal{T}$ be a one-dimensional tiling. Recall that we identify $\mathcal{T}$ with a bi-infinite word α over a finite alphabet Σ and that $L(\mathcal{T})$, the set of all finite factors of α, is a language over Σ, whose elements represent the pattern classes of the tiling. Thus, each non-zero element of the tiling semigroup $S(\mathcal{T})$ can be represented as a finite string with underlying word belonging to $L(\mathcal{T})$ and two distinguished positions to mark the in- and out-tiles. Identifying the in-tile by a grave accent, the out-tile by an acute accent, and the coinciding in-tile and out-tile by a check, we get that the non-zero elements of $S(\mathcal{T})$ can be represented as:

$$x\grave{a}y\acute{b}z, \text{ with } a, b \in \Sigma,\ x, y, z \in \Sigma^*, \text{ and } xaybz \in L(\mathcal{T})\,, \text{ or}$$

$$x\acute{a}y\grave{b}z, \text{ with } a, b \in \Sigma,\ x, y, z \in \Sigma^*, \text{ and } xaybz \in L(\mathcal{T})\,, \text{ or}$$

$$x\breve{a}y, \text{ with } a \in \Sigma,\ x, y \in \Sigma^*, \text{ and } xay \in L(\mathcal{T})\,,$$

according to whether the in-tile comes before, after, or in the same tile as the out-tile, respectively. Given a non-zero element s, we denote its underlying word by $\underline{s}$.

Translating the tiling semigroup operation into this representation gives us that the product of two non-zero elements s and t is non-zero when $\underline{s}$ and $\underline{t}$ match and the resulting word belongs to $L(\mathcal{T})$, when the out-letter of s and the in-letter of t are superimposed; the resulting string inherits the in-letter of s and the out-letter of t.

Example 3.2.4. Consider the one-dimensional tiling $\mathcal{T}$ associated with the bi-infinite word over $\{a, b\}$

$$\dots a\,b\,b\,a\,b\,b\,a\,b\,b\,\dots\;.$$

Then $\grave{a}\acute{b}$, $\acute{a}\grave{b}$, $\check{b}a$ and $a\acute{b}\grave{b}$ are non-zero elements of $S(\mathcal{T})$, and:

- $a\acute{b}\grave{b} \cdot \grave{a}\acute{b} = 0$, as the out-letter of $a\acute{b}\grave{b}$, which is (its first) b, and the in-letter of $\grave{a}\acute{b}$, which is a, are different;

- $\grave{a}\acute{b} \cdot a\acute{b}\grave{b} = 0$, as the words ab and abb do not match when we superimpose the out-letter of $\grave{a}\acute{b}$, which is b, which the in-letter of $\acute{b}\grave{b}$, which is its second b;

- $\grave{a}\acute{b} \cdot \check{b}a = 0$, as the resulting word, aba, does not belong to $L(\mathcal{T})$;

- $\check{b}a \cdot a\acute{b}\grave{b} = a\acute{b}\grave{b}a$;

- $\grave{a}\acute{b} \cdot \acute{a}\grave{b} = \breve{a}b$;

- $\check{b}a \cdot \check{b}a = \check{b}a$;

As noted in Section 3.2, we have that $[a, A, b]^{-1} = [b, A, a]$ and that $[a, A, b]$ is idempotent if and only if $a = b$, for any non-zero element $[a, A, b]$ in an arbitrary tiling semigroup. Therefore, in the representation of one-dimensional semigroups that we have just described, the inverse of a non-zero element is found by simply interchanging the accents and the non-zero idempotents are the elements of the form $x\breve{a}y$ (with $a \in \Sigma$, $x, y \in \Sigma^*$, and $xay \in L(\mathcal{T})$).

This representation of one-dimensional tilings semigroups is due to Lawson [32], [33].

Inverse semigroups associated with factorial languages

As noted in the previous section, the tiling language of a one-dimensional tiling is always a factorial language. Starting from an arbitrary factorial language L (over a finite alphabet Σ), we can construct a semigroup in the same way a one-dimensional tiling semigroup is constructed from the tiling language: the non-zero elements are the doubly marked strings whose underlying word belongs to L and the operation is defined in exactly the same way. We thus arrive at an inverse semigroup with zero, which we denote by $S(L)$ and which coincides with $S(\mathcal{T})$ when $L = L(\mathcal{T})$.

Definition 3.2.5. Let L be a factorial language over a finite alphabet. We call $S(L)$ the *inverse semigroup associated with* L.

We note that inverse semigroups associated with factorial languages are indeed a generalization of one-dimensional tiling semigroups, for an arbitrary factorial language needs not be a tiling language. As noticed in Section 1.3, any language of the factors of a bi-infinite word, that is, any tiling language L is not only factorial but also extensible and such that

$$\forall u, v \in L\ \exists w \in L,\ u, v \in F(w)\,, \tag{3.1}$$

and Examples 1.3.2 and 1.3.3 show that a factorial language need not satisfy these properties. Therefore, the difference between a one-dimensional tiling semigroup $S(L(\mathcal{T}))$ and a semigroup $S(L)$ associated with a factorial language L is that L does not have to come from a one-dimensional tiling as its tiling language.

It is easy to see that inverse semigroups associated with factorial languages enjoy of all the properties of one-dimensional tiling semigroups mentioned so far. In the next section, we will review the properties of tiling semigroups (and of inverse semigroups associated with factorial languages) that are known in the literature.

3.3 Tiling semigroups in the literature

In this section we review the material known on this subject prior to our investigations, as well as the contributions given recently by other authors, in the context of inverse semigroup theory.

As already mentioned, tiling semigroups were originally introduced by Kellendonk under the form of an almost-groupoid [24] and then adapted by Lawson to the context of inverse semigroup theory [32]. Here, Lawson establishes the first known properties of tiling semigroups: they are combinatorial, completely semisimple, the Green's relation $\mathcal{D}$ is characterized, and, most remarkably, they are E^*-unitary. Lawson pays special attention to one-dimensional tiling semigroups, and using its language representation proves a fundamental result that states that the tiling semigroup $S(\mathcal{T})$, of a one-dimensional tiling $\mathcal{T}$ over Σ, divides the McAlister semigroup $M_{|\Sigma|}$. Recall that, given a positive integer n, the *orthogonally generated inverse semigroup M_n on n generators*, also called *McAlister semigroup on n generators*, is the semigroup with zero generated by the elements $x_1, \ldots, x_n$, say, where $x_i {x_j}^{-1} = {x_i}^{-1} x_j = 0$ for all $i \neq j$. In the case of $n = 1$, the zero is omitted and M_1 is precisely the free inverse semigroup on one generator. These semigroups were first considered and studied by Lawson in [31]. There, Lawson begins by defining M_n as the set of triples $(u, v, w) \in \Sigma^* \times \Sigma^+ \times \Sigma^*$, with u a prefix of v and w a suffix of v, where Σ is an alphabet with n letters, under the operation defined on the non-zero elements by

$$(u,v,w)(r,s,t) = \begin{cases} \big(((rw)v^{-1})\,u, ((rw)v^{-1})\,v\,((rw)^{-1}s), t\,(s^{-1}(rw))\big), \\ \qquad\qquad\qquad\qquad \text{if } \Sigma^*(rw) \cap \Sigma^* v \neq \emptyset \text{ and } (rw)\Sigma^* \cap s\Sigma^* \neq \emptyset \\ 0, \qquad\qquad\qquad\quad \text{otherwise,} \end{cases}$$

where, for all $x, y \in \Sigma^*$,

$$y^{-1}x = \begin{cases} z, & \text{if } x = yz \\ 1, & \text{otherwise} \end{cases} \qquad \text{and} \qquad xy^{-1} = \begin{cases} z, & \text{if } x = zy \\ 1, & \text{otherwise.} \end{cases}$$

In order to show that such a semigroup admits the desired presentation, Lawson proves that the non-zero elements of

$$Inv_0 \left\langle x_i \; (x_i \in \Sigma) \mid x_i^{-1}\, x_j = 0 = x_i\, x_j^{-1} \; (\text{for } i \neq j) \right\rangle$$

are those which can be written in the for $u^{-1}vw^{-1}$, with $u, v, w \in \{x_i \colon i \in [n]\}^*$, $v \neq 1$, u a prefix of v and w a suffix of v. With respect to tiling semigroups, in [32], Lawson used his original representation of a McAlister semigroup to prove that

Theorem 3.3.1 (Theorem 9.5.5, [32])**.** *Let m be a positive integer and Σ an alphabet with m letters. Then the mapping $\phi \colon S(\Sigma^+) \to M_m$, defined by*

$$\begin{cases} 0\,\phi = 0, \\ x\breve{a}y\,\phi = (x, xay, ay), \\ x\grave{a}y\acute{b}z\,\phi = (x, xaybz, bz), \\ x\acute{a}y\grave{b}z\,\phi = (xay, xaybz, aybz)\,, \end{cases}$$

is an injective homomorphism, with

$$S(\Sigma^+)\phi = \{0\} \cup \{(u,v,w) \colon u \neq v \text{ and } w \neq 1\} \subseteq M_m\,.$$

At this stage, Lawson does not realize $S(\Sigma^+)$ as being the inverse semigroup associated with a factorial language but rather as an "abstract" semigroup constructed from the alphabet Σ, which he calls the *Kachel semigroup on m letters* and denotes by K_m. Moreover,

Theorem 3.3.2 (Theorem 9.5.6, [32]). *The tiling semigroup of a one-dimensional tiling $\mathcal{T}$ over Σ, where Σ contains m letters, is isomorphic to the Rees factor semigroup $S(\Sigma^+)/I$, where I is the ideal of $S(\Sigma^+)$*

$$I = \{0\} \cup \left\{ s \in S(\Sigma^+) \colon \text{ the underlying word of } s \text{ does not belong to } L(\mathcal{T}) \right\} .$$

In particular, every one-dimensional tiling semigroup over an alphabet with m letters divides M_m.

$$\begin{array}{ccc} S(\Sigma^+) & \hookrightarrow & M_m \\ \downarrow & & \\ S(\Sigma^+)/I \simeq S(\mathcal{T}) & & \end{array}$$

Kellendonk and Lawson [26] provided a categorical basis for the tiling semigroup, in terms of a group acting partially and without fixed points on a category and used the fact that the tiling semigroup of the simplest one-dimensional tiling, namely

$$\cdots a\, a\, a\, a\, a\, a \cdots ,$$

is no other than the free inverse monoid on one generator (with zero adjoined), to show that one-dimensional tiling semigroups are in fact strongly E^*-unitary.

In [28], the same authors present the notion of the *universal group* of a set equipped with an associative partial binary operation, which, when applied to a semigroup S with zero, consists of the quotient of the free group FG_S on S by the congruence on FG_S generated by the pairs $((ab)\iota, a\iota\, b\iota)$, with $a, b \in S$ such that $ab \neq 0$, where $\iota \colon S \to FG_S$ denotes the inclusion map. They then show that the universal group of a one-dimensional tiling semigroup is isomorphic to a free group, the number of generators of which can be read off from the tiling.

In [33], Lawson considers one-dimensional tiling semigroups, or, more precisely, inverse semigroups associated to factorial languages as described in the previous subsection. He finds a very useful canonical normal form for the non-zero elements of such semigroups, gives an universal characterization for these semigroups, as the initial object in a category of homomorphisms, and shows that, if L is not only factorial but also extensible, $S(L)$ can be represented faithfully on the shift space constructed from L. To do so, Lawson uses a construction that again involves McAlister semigroups.

Lawson's canonical form for the non-zero elements of a one-dimensional tiling semigroup, or of the non-zero elements of an inverse semigroup associated with a factorial language, will be very useful in the next chapter. For that reason, we include a summary of this result. Let L be a factorial language over an alphabet Σ. Given a non-zero element $s \in S(L)$ with, say, underlying word $a_1 \ldots a_k$ (with $k \geqslant 2$), in-letter a_i and out-letter a_j, we have

$$s = \begin{cases} (\grave{a}_1 \ldots \acute{a}_i)^{-1}\ \grave{a}_1 \ldots \acute{a}_k\ (\grave{a}_j \ldots \acute{a}_k)^{-1}, & \text{if } i \neq 1 \text{ and } j \neq k \\ \breve{a}_1\, \grave{a}_1 \ldots \acute{a}_k\ (\grave{a}_j \ldots \acute{a}_k)^{-1}, & \text{if } i = 1 \text{ and } j \neq k \\ (\grave{a}_1 \ldots \acute{a}_i)^{-1}\ \grave{a}_1 \ldots \acute{a}_k\, \breve{a}_k, & \text{if } i \neq 1 \text{ and } j = k \\ \breve{a}_1\, \grave{a}_1 \ldots \acute{a}_k\, \breve{a}_k, & \text{if } i = 1 \text{ and } j = k\,, \end{cases}$$

so that every non-zero element of $S(L)$ whose underlying word has length at least 2 can be written in a unique way in the form $u^{-1}\, v\, w^{-1}$, with u, v, and w non-zero elements of the subsemigroup

$$\check{C}(L) = \{\grave{a}_1 \ldots \acute{a}_k \colon a_1 \ldots\, a_k \in L,\ k \geqslant 2\} \cup \{\breve{a} \colon a \in L\} \cup \{0\}$$

of $S(L)$ and $v \in \big(u\, \check{C}(L)\big) \cap \big(\check{C}(L)\, w\big)$. The last condition simply states that the underlying word of u is a non-empty prefix of the underlying word of v and that the underlying word of w is a non-empty suffix of the underlying word of v. Since the non-zero elements whose underlying word consists only of a single letter have a unique representation, this result establishes then a canonical form for the elements of $S(L)$.

After Kellendonk and Lawson, several authors have studied tiling semigroups from the point of view of inverse semigroups, for example Dombi and Gilbert [12], [14], McAlister [40], [41], McAlister and FS [43], [44], FS [58], and Zhu [61].

In [40], McAlister characterized one-dimensional tiling semigroups in a new manner as being isomorphic to the face of a certain set of idempotents of the free inverse semigroup with zero adjoined. This description illuminates the connection between tiling semigroups and Munn trees. It was also from here on that the current, more semigroup-like notation for the non-zero elements of a tiling semigroup was introduced: McAlister denotes a non-zero element as $[a_i, A, a_o]$ instead of Lawson's $[a_o, A, a_i]$, with the operation defined as we have in Section 3.2, instead of Lawson's original definition as $[a_o, A, a_i][b_o, B, b_i] = [a_o + x, (A + x) \cup (B + y), b_i + y]$ if there exist translations x and y of $\mathbb{R}^n$ such that $(A + x)$ and $(B + y)$ are patterns in the tiling and $a_i + x = b_o + y$ (and zero otherwise). Both these definitions are, of course, equivalent, for their difference is one of terminology: which tile to call in-tile and which to call out-tile. (In fact, straightforward checking shows that we have an isomorphism from Lawson's tiling semigroup to McAlister's tiling semigroup defined on the non-zero elements by $[a_o, A, a_i]_L \phi = [a_i, A, a_o]_M^{-1} = [a_o, A, a_i]_M$, where the self-explained subscripts L and M simply aim at avoiding confusion.) This is the notation chosen for this work; accordingly, the in-tile and the out-tile are represented, in the one-dimensional case, by a grave accent and an acute accent, respectively, which is the opposite of the notation adopted by Lawson in [32] and [33] but agrees with that of Kellendonk [24].

In [41], McAlister generalized Kellendonk and Lawson's result, that states that one-dimensional tiling semigroups are strongly E^*-unitary, to arbitrary tiling semigroups and used this fact to provide an alternative description of tiling semigroups in a way which avoids the explicit use of equivalence classes.

Using Kellendonk and Lawson's categorical approach, in [61] Zhu gives a complete description the Green's relations on a tiling semigroup S and studies other relations and congruences defined on S. In particular, Zhu shows that a property of the tiling, namely the notion of patterns having the same period, yields a binary relation on the tiling semigroup that turns out to be an idempotent-pure congruence.

In our McS. thesis [58], we give an account on the theory on tiling semigroups. Along with a detailed description of the results contained in the papers mentioned above, the proof of a few other properties can be found, as, for instance, an explicit characterization of the natural partial order (already alluded to in [32]) and the fact that n-dimensional tiling semigroups, with $n \geqslant 2$, need not be F^*-inverse, although one-dimensional tiling semigroups are always strongly F^*-inverse.

In [43], together with McAlister, we further the study of one-dimensional tiling semigroups as a particular case of the inverse semigroup associated with a factorial language. For such a semigroup, we give a description as a P^*-semigroup and a presentation. Also we investigate different classes of languages with respect to producing finite or infinitely presented semigroups. These results can be found in Chapter 5, Section 5.1 and Chapter 6, Section 6.1.

In [44], the same authors generalized to a particular class of n-dimensional tilings two important results on one-dimensional tiling semigroups, namely: the description of the semigroup as the inverse semigroup associated with a factorial language and its representation as a P^*-semigroup. In addition, it is proved that these semigroups are always infinitely presented as inverse semigroups and a necessary and sufficient condition for such tiling semigroups to be isomorphic is given. These results can be found in Chapter 4, Sections 4.1 and 4.2, Chapter 5, Section 5.2, Chapter 6, Section 6.2, and Chapter 7.

In [12], Dombi and Gilbert characterize the structure of tiling semigroups of one-dimensional periodic tilings, using the representation of one-dimensional tiling semigroups given in Theorem 3.3.2. Their main result is a complete description of such a semigroup as a subsemigroup of a Rees factor semigroup of a semidirect product in which the free monogenic monoid acts on the subsets of $\mathbb{Z}_m$, where m is the length of the period of the tiling. This work is a beautiful and deep insight into the strongly E^*-unitary property of tiling semigroups.

In [14], Dombi and Gilbert construct a strongly F^*-inverse cover of arbitrary n-dimensional tiling semigroups with $n \geqslant 2$ (which, as already mentioned, need not be F^*-inverse, contrary to one-dimensional tiling semigroups) and show that it admits a decomposition as an HNN* extension of its semilattice of idempotents. For this construction, the notion of path extension of a tiling semigroup is introduced.

Finally, a reference should be made a paper by Masuda and Morita [37]. Although their work does not fit in the context of inverse semigroup theory, it is an algebraic approach to tilings. The authors investigate a bialgebra constructed from the tiling semigroup (more precisely, from the tiling semigroup with an adjoined identity). Their main result characterizes under which conditions tilings of the real line give rise to isomorphic associated bialgebras, for which the notion of locally nondistinguishable tilings is introduced.

3.4 Hypercubic tilings and hypercubic tiling semigroups

One of the main difficulties when dealing with arbitrary dimensional tiling semigroups is that they do not come as generalizations of one-dimensional tiling semigroups or even as semigroup constructions with these as building blocks. In an attempt to consider an intermediate case between one-dimensional tiling semigroups and arbitrary dimensional tiling semigroups, we engaged in the study of what we called hypercubic tilings. To the best of our knowledge, tiling semigroups of hypercubic tilings have not been studied before.

In this final section, we define the class of n-dimensional hypercubic tilings, introduce the concept of coloured subset of $\mathbb{Z}^n$ to more conveniently express the elements of the tiling semigroup associated with such a tiling, and find a generating set for the semigroup.

Recall that the *n-dimensional hypercube (of side 1)* is the set $[0,1]^n$, that is

$$H_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon 0 \leqslant x_i \leqslant 1 \text{ for all } i \in [n]\} \ .$$

Definition 3.4.1. An *n-dimensional hypercubic tiling* is a tiling of $\mathbb{R}^n$ by hypercubes of side 1 (that is, translations of H_n), possibly coloured, and aligned in such a way that the centre of each tile belongs to $\mathbb{Z}^n$.

For $n = 1$, the hypercubes of length 1 are precisely the intervals of length 1. Although this may seem like a strong restriction to impose on one-dimensional tilings, that is not the case. On the contrary, since we only consider finite type tilings, a one-dimensional tiling is a countable collection of closed, bounded intervals of $\mathbb{R}$, whose lengths must belong to a finite set. This implies that all one-dimensional tilings can be thought of as one-dimensional hypercubic tilings: instead of regarding the tiling as a collection of intervals of (possibly) different lengths, we assign a colour to each interval length occurring in the tiling and consider the tiles as being coloured intervals with length 1. This is, in fact, exactly what is carried on when we identify a one-dimensional tiling with a bi-infinite word over a finite alphabet. Letters from this alphabet are just what, from the point of view of hypercubic tilings, we think of as colours. Therefore, the class of one-dimensional hypercubic tilings can be identified with the class of one-dimensional tilings.

For $n = 2$, the hypercubes of length 1 are now squares of side 1. Thus, a two-dimensional hypercubic tiling is collection of squares, aligned both vertical and horizontally, possibly single coloured (Figure 3.5), or either with some inner structure among the colours (Figure 3.6) or not (Figure 3.7).

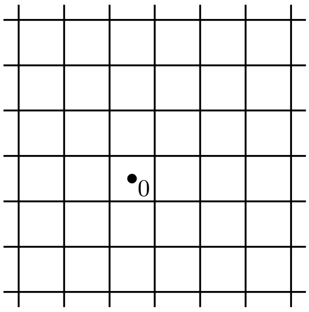


Figure 3.5: A two-dimensional single coloured hypercubic tiling

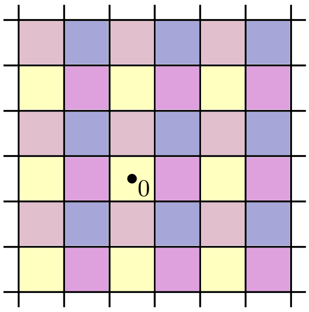


Figure 3.6: A two-dimensional “regular” hypercubic tiling

Notice that, regardless of the dimension, the requirement on the alignment of the centre of the tiles means that adjacent tiles share a common face.

For convenience, we will refer to the tiling semigroup of an n-dimensional hypercubic tiling as an *n-dimensional hypercubic tiling semigroup*.

In the one-dimensional case, Lawson's identification of the tiling with a bi-infinite word allows for a much more convenient representation of the tiling semigroup. This is also true for hypercubic tilings. Their special structure also lets us set up an identification that will prove itself useful. By definition of hypercubic tiling, the centre of each tile belongs to $\mathbb{Z}^n$. Therefore, an n-dimensional hypercubic tiling $\mathcal{T}$ whose tiles are coloured with colours from a finite set Σ can be identified with a map $\tau\colon \mathbb{Z}^n \to \Sigma$ in the obvious way: for each $x \in \mathbb{Z}^n$, the element

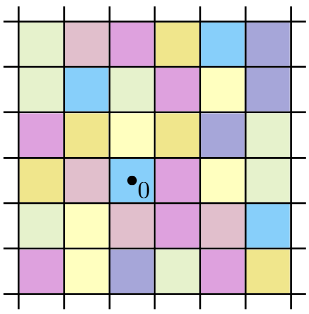


Figure 3.7: A two-dimensional “disorganized” hypercubic tiling

$\tau(x)$ of Σ is the colour of the hypercube in the tiling whose centre is x. To indicate that we are making such an identification, we write $\mathcal{T} = (\mathbb{Z}^n, \tau)$. Conversely, each map $\tau\colon \mathbb{Z}^n \to \Sigma$ from $\mathbb{Z}^n$ onto a finite set Σ gives rise to a unique n-dimensional hypercubic tiling. To take advantage of this observation, we define the following set of notions.

Recall that a *Cayley graph of a group G* is a directed labelled graph whose vertices are in one-to-one correspondence with the elements of the group and whose edges are labelled by the group generators. Since an arbitrary group can be expressed in terms of different generators, a group does not have a unique Cayley graph. For that reason, it is useful to define, for a group G with group presentation $Gp\,\langle X|R\rangle$, the *Cayley graph of the presentation* $Gp\,\langle X|R\rangle$ as the graph $\Gamma(X;C)$ with vertex set G and edges (g, gx) with $g \in G$ and $x \in X$.

Example 3.4.2. In Figure 3.8, we have the Cayley graph of $\mathbb{Z}^2$ with respect to the usual group presentation $Gp\,\langle a, b \mid a + b = b + a\rangle$.

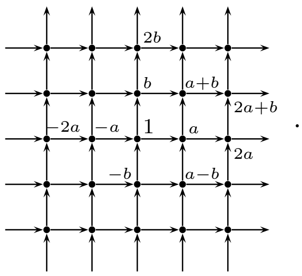


Figure 3.8: The Cayley graph of $\mathbb{Z}^2$ with respect to the usual group presentation

We say that two vertices x and y of a directed graph are *adjacent* if either (x, y) or (y, x) are edges in the graph. Identifying $\mathbb{Z}^n$ with its Cayley graph of the usual group presentation $Gp\,\langle a_1, \ldots, a_n \mid a_i + a_j = a_j + a_i\,(i, j \in [n])\rangle$, we have, in particular, that two vertices x and y of $\mathbb{Z}^n$ are adjacent if $x - y$ is either a standard basis vector or the negative of a standard basis vector of $\mathbb{Z}^n$.

Let n be an arbitrary positive integer and Σ a finite set.

Definition 3.4.3. By a *Σ-coloured subset of* $\mathbb{Z}^n$ we shall mean a pair (A, α) where A is a non-empty subset of $\mathbb{Z}^n$ and α is a map from A to Σ. We say that (A, α) is *finite* if A is finite; we say that it is *connected* if A is the vertex set of a connected subgraph of the Cayley graph of $\mathbb{Z}^n$ with respect to the usual set of generators.

Because the domain of the map α is the set A, the "A" in the notation (A, α) may seem superfluous. However, instead of omitting it, we will in fact sometimes omit reference to the map, in situations when it is determined by the context. Likewise, we may sometimes omit reference to the set Σ provided there is no danger of confusion.

There is an obvious partial order on the set of Σ-coloured subsets of $\mathbb{Z}^n$, namely the one defined by the rule

$$(A, \alpha) \leqslant (B, \beta) \Leftrightarrow A \subseteq B \text{ and } \beta(a) = \alpha(a) \text{ for each } a \in A, \text{ that is, } \alpha = \beta|_A\,,$$

for all Σ-coloured subsets (A, α) and (B, β) of $\mathbb{Z}^n$. If $(A, \alpha) \leqslant (B, \beta)$, we say that (A, α) *is a Σ-coloured subset of* (B, β).

The group $\mathbb{Z}^n$ acts in a natural way — by translation — on the subsets of $\mathbb{Z}^n$: for every $u \in \mathbb{Z}^n$ and for every subset A, set $A + u$ to be the subset $\{a + u \colon a \in A\}$ of $\mathbb{Z}^n$. Of course, finiteness and connectedness are preserved. Moreover, this action can easily be made to preserve colours as well. Let $(A, \alpha) + u = (A + u, \delta)$ be the Σ-coloured subset of $\mathbb{Z}^n$ with colour map $\delta \colon A + u \to \Sigma$ defined by $\delta(a + u) = \alpha(a)$ for each $a \in A$, for every $u \in \mathbb{Z}^n$ and Σ-coloured subset (A, α) of $\mathbb{Z}^n$. Observe that $\delta(a+u) = \alpha(a)$ is equivalent to $\delta(x) = \alpha(x-u)$, for each $x \in A + u$.

In this way, an action on the set of all Σ-coloured subsets of $\mathbb{Z}^n$ is defined that preserves finiteness, connectedness, and colours.

The following partially defined operation on the Σ-coloured subsets of $\mathbb{Z}^n$ will be instrumental in our use of these objects. If (A, α) and (B, β) are Σ-coloured subsets of $\mathbb{Z}^n$ such that $\alpha|_{A\cap B} = \beta|_{A\cap B}$, let $(A, \alpha) \cup (B, \beta) = (A \cup B, \gamma)$ be the Σ-coloured subset of $\mathbb{Z}^n$ with colour map given by

$$\gamma(x) = \begin{cases} \alpha(x) & \text{if } x \in A \\ \beta(x) & \text{if } x \in B\,. \end{cases}$$

Evidently, this operation needs not be defined for arbitrary Σ-coloured subsets of $\mathbb{Z}^n$, but it is certainly well-defined when $\alpha|_{A\cap B} = \beta|_{A\cap B}$.

Remark 3.4.4. For a more suggestive notation, usually we will denote by $\alpha + u$ the map δ and by $\alpha \cup \beta$ the map γ is the previous paragraphs.

Fix an n-dimensional hypercubic tiling $\mathcal{T} = (\mathbb{Z}^n, \tau)$ with colours in Σ. Because each finite, connected Σ-coloured subset (A, α) of $(\mathbb{Z}^n, \tau)$ (not simply of $\mathbb{Z}^n$, note) corresponds to a unique pattern in the tiling, we can identify the non-zero elements of the tiling semigroup $S(\mathcal{T})$ with the equivalence classes defined by the following equivalence relation: for all (A, α) and (C, γ) finite, connected Σ-coloured subsets of $(\mathbb{Z}^n, \tau)$, $a, b \in A$, and $c, d \in C$,

$$(a, (A, \alpha), b) \sim (c, (C, \gamma), d) \Leftrightarrow \exists\, x \in \mathbb{Z}^n \text{ such that } (A, \alpha) + x = (C, \gamma), a + x = c \text{ and } b + x = d\,,$$

In order to obtain the tiling semigroup $S(\mathcal{T})$ of the n-dimensional hypercubic tiling $\mathcal{T}$, we equip this set with the following multiplication, which clearly coincides with that of $S(\mathcal{T})$: every product involving zero yields zero and, given $[a, (A, \alpha), b], [c, (C, \gamma), d] \in S(\mathcal{T})$, we set

$$[a, (A, \alpha), b][c, (C, \gamma), d] = [a + x, ((A, \alpha) + x) \cup ((C, \gamma) + y), d + y]\,,$$

in case $(\alpha + x)|_{(A+x)\cap(C+y)} = (\gamma + y)|_{(A+x)\cap(C+y)}$ for some translations $x, y \in \mathbb{Z}^n$ such that $b + x = c + y$, and zero otherwise.

Remark 3.4.5. Before proceeding, we adopt the following abuse of notation. When $(A, \alpha) \leqslant (B, \beta)$, then the map α is completely determined by β for it is the restriction of β to A, which is a subset of the domain of β. In particular, if $(A, \alpha) \leqslant (\mathbb{Z}^n, \tau)$, then $\alpha = \tau|_A$. For this reason, we will write A instead of (A, α) — and, thus, $A \leqslant (\mathbb{Z}^n, \tau)$ instead of $(A, \alpha) \leqslant (\mathbb{Z}^n, \tau)$ and $A + x$ instead of $(A, \alpha) + x$.

Also, we will write $A \cup C$ instead of $A \vee C$, but bearing in mind that we think of A and C as Σ-coloured finite, connected subsets of $(\mathbb{Z}^n, \tau)$.

To conclude this section, and chapter, we turn the question of finding a generating set for an n-dimensional hypercubic tiling. This will be of use in Chapters 5, 6 and 7.

Generators for hypercubic tiling semigroups

For motivation, we begin with the one-dimensional case, and, as usual, we consider the more general case of inverse semigroups associated with factorial languages. So let L be a factorial language over an alphabet Σ. Recall from Section 3.3, that there is a canonical form for the non-zero elements of $S(L)$: these are either one-letter idempotents or elements whose underlying word has at least two letters, all of which can be written in a unique way in the form $(\grave{a}_1 \ldots \acute{a}_i)^{-1} \grave{a}_1 \ldots \acute{a}_k (\grave{a}_1 \ldots \acute{a}_j)^{-1}$, for some $a_1 \ldots a_k \in L$ and $i, j \in [k]$, as shown by Lawson in [33]. Since

$$\grave{a}_1 \ldots \acute{a}_k = \grave{a}_1\acute{a}_2\, \grave{a}_2\acute{a}_3\, \ldots\, \grave{a}_{k-1}\acute{a}_k$$

(note that the fact that L is factorial implies that $a_1a_2, a_2a_3, \ldots, a_{k-1}a_k \in L$), we conclude that $S(L)$ is generated, as an inverse semigroup with zero, by its one-letter idempotents (that is, the elements of the form $\breve{a}$, with $a \in \Sigma$ or, equivalently, $a \in L$), and its two-letter elements of the form $\grave{a}_i\acute{a}_j$, with $a_ia_j \in L$.

Now let $\mathcal{T} = (\mathbb{Z}^2, \tau)$ be a two-dimensional hypercubic tiling with colours in an alphabet Σ. We claim that $S(\mathcal{T})$ is generated, as an inverse semigroup with zero, by the $|\Sigma|$ single-tile idempotents e_i, with $i \in \Sigma$, and the two-tile elements $a_{(i,j)}$ and $b_{(i,j)}$, with $i, j \in \Sigma$, where the vector from the in-tile to the out-tile of $a_{(i,j)}$ is the standard basis vector $(1, 0)$, the vector from the in-tile to the out-tile of $b_{(i,j)}$ is the standard basis vector $(0, 1)$, and such that the underlying finite, connected Σ-coloured subset is a Σ-coloured subset of $\mathcal{T} = (\mathbb{Z}^2, \tau)$.

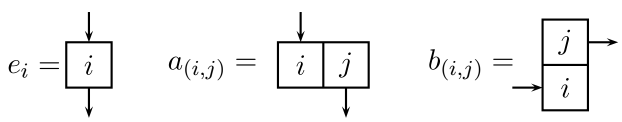


Figure 3.9: The generators of a two-dimensional hypercubic tiling

Lemma 3.4.6. *Let $\mathcal{T}$ be a two-dimensional hypercubic tiling with colours in an alphabet Σ and $[a, (A, \alpha), b]$ a non-zero element from $S(\mathcal{T})$ with $|A| \geqslant 2$. Then $[a, (A, \alpha), b]$ can be written as a product of elements in the set*

$$X = \{a_{(i,j)}, a_{(i,j)}^{-1}, b_{(i,j)}, b_{(i,j)}^{-1} \in S(\mathcal{T}) \colon i, j \in \Sigma\}.$$

Proof. Let $[a,(A,\alpha),b] \in S(\mathcal{T})$ with $|A| \geqslant 2$. Since, by definition, A is finite and the vertex set of a connected subgraph of $\mathbb{Z}^2$, then $A = \bigcup_{k=1}^{m}\{x_k, x_{k+1}\}$, for some $x_1, \ldots, x_{m+1} \in \mathbb{Z}^2$ (not necessarily distinct, of course) such that $\{x_k, x_{k+1}\}$ is a two-element connected set for all $k \in [m]$. But then each vector $x_{k+1} - x_k$ is either a standard basis vector of $\mathbb{Z}^2$ or the negative of a standard basis vector, and so $[x_k, (\{x_k, x_{k+1}\}, \alpha|_{\{x_k,x_{k+1}\}}), x_{k+1}]$ is one of the elements $a_{(\alpha(x_k),\alpha(x_{k+1}))}$, $a^{-1}_{(\alpha(x_{k+1}),\alpha(x_k))}$, $b_{(\alpha(x_k),\alpha(x_{k+1}))}$ or $b^{-1}_{(\alpha(x_{k+1}),\alpha(x_k))}$. Since a and b belong to A, we may assume without loss of generality that $x_1 = a$ and $x_m = b$. Thus,

$$[a,(A,\alpha),b] = \prod_{k=1}^{m} [x_k, (\{x_k, x_{k+1}\}, \alpha|_{\{x_k,x_{k+1}\}}), x_{k+1}] ,$$

and we conclude that $[a,(A,\alpha),b]$ can be written as a product of elements in X. $\square$

In view of this result, the following is immediate:

Corollary 3.4.7. *Let $\mathcal{T}$ be a two-dimensional hypercubic tiling with colours in an alphabet Σ. Then the elements e_i, with $i \in \Sigma$, and $a_{(i,j)}$ and $b_{(i,j)}$, with $i, j \in \Sigma$, of $S(\mathcal{T})$ described above generate $S(\mathcal{T})$ as an inverse semigroup with zero.*

It is now easy to see that, in general,

Proposition 3.4.8. *The tiling semigroup $S(\mathcal{T})$ of an n-dimensional hypercubic tiling $\mathcal{T} = (\mathbb{Z}^n, \tau)$ with colours from a set Σ, is generated, as an inverse semigroup with zero, by the $|\Sigma|$ single-tile idempotents and the following two-tile elements $a_{1(i,j)}, a_{2(i,j)}, \ldots$ and $a_{n(i,j)}$ (with $i, j \in \Sigma$): for each $k \in [n]$, $a_{k(i,j)} = [a,(A,\alpha),b]$ with $A = \{a,b\}$, (A,α) is a Σ-coloured subset of $(\mathbb{Z}^n, \tau)$, $\alpha(a) = i$, $\alpha(b) = j$ and $b - a$ is the k^{th} standard basis vector of $\mathbb{Z}^n$.*

Note that, since Σ is a finite set, there are $|\Sigma|$ single-tile idempotent generators and at most $n\,|\Sigma|^2$ two-tile generators, depending on the arrangement of colours in $\mathcal{T}$, in the generating set from the previous proposition.

So far, we have defined n-dimensional hypercubic tiling with colours from a set Σ, identified such a tiling with a so-called Σ-coloured subset $(\mathbb{Z}^n, \tau)$ of $\mathbb{Z}^n$ read off from the tiling and the patterns in the tiling with the finite, connected Σ-coloured subsets of $(\mathbb{Z}^2, \tau)$, used this identification to write the non-zero elements and the operation of the tiling semigroup in terms of these objects, and described a generating set for $S(\mathcal{T})$. In the next chapter, we will see how to define an appropriate notion of language of a hypercubic tiling and how to use it to build a language representation of the semigroup associated with the tiling that generalizes the language representation of one-dimensional tiling semigroups.

Chapter 4

Language representation of hypercubic tiling semigroups

All investigations into one-dimensional tiling semigroups, namely those of Lawson's [33] and Dombi and Gilbert's [12], [13], as well as our own [43], relied entirely on the representation of the semigroup as the inverse semigroup associated with a factorial language, the language of the tiling. To find such a convenient description for hypercubic tiling semigroups was, thus, a natural starting point. It turns out that hypercubic tiling semigroups do indeed admit a fairly neat representation in this way, perhaps even to the point of hidding how surprising it is. Equally remarkable is how, as we will see in Chapter 5, with this representation, a description of hypercubic tiling semigroups as P^*-semigroups can be found that matches exactly with the one for one-dimensional tiling semigroups.

The chapter is organized as follows. In Section 4.1 we will generalize the notion of language of a one-dimensional tiling to arbitrary dimensional hypercubic tilings; more precisely, we will generalize the notion of n-dimensional factorial language, of which the former is a particular case. To do this, we will use the important concept of coloured subset of $\mathbb{Z}^n$ developed in Section 3.4. In Section 4.2, we construct an inverse semigroup associated with an n-dimensional factorial language and prove that, in the case of a language associated with an n-dimensional hypercubic tiling, that semigroup and the tiling semigroup are isomorphic. In addition, we will show that it coincides with Lawson's language representation of one-dimensional tiling semigroup [32] when $n = 1$. Motivated by the form of the multiplication in the language representation, we investigate in Section 4.3 the connection between hypercubic tiling semigroups and Bruck-Reilly extensions by means of the construction developed in Chapter 2.

4.1 n-dimensional factorial languages and the language of an n-dimensional hypercubic tiling

Recall from the subsection of Section 3.1 concerning one-dimensional tilings, that, in this case, the alphabet of the tiling language corresponds to the set of lengths of the intervals in the tiling or, in terms of hypercubic tilings, to the set of their colours; as to the words, they correspond to the factors of the bi-infinite word which we identify with the tiling, that is, the patterns of the tiling identified up to translation. In order to define a corresponding notion of tiling language for an n-dimensional hypercubic tiling, we need to undertake a similar procedure: store the collection of patterns that occur in the tiling, regardless of their position, in an efficient way. Notice that the comparison of patterns is two-fold: both the geometric display of tiles and the display of the colours must agree. (In fact, the comparison of words may be seen as being two-fold as well, for two words are equal if and only if they have the same length and the same sequence of letters.) Consequently, our notion of language must allow us to compare shapes, and colours within shapes.

From what we have seen in Section 3.4, comparing patterns in an n-dimensional hypercubic tiling $\mathcal{T} = (\mathbb{Z}^n, \tau)$ with colours from a set Σ amounts to comparing finite, connected Σ-coloured subsets of $(\mathbb{Z}^n, \tau)$. With this in mind, we consider the following well-known order in $\mathbb{Z}^n$ (the lexicographic order comparing from right to left):

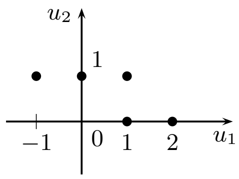
Lemma 4.1.1. *The binary relation defined by*

$$x \leqslant y \Leftrightarrow x = y \text{ or there exists } 1 \leqslant i \leqslant n \text{ such that } x_i < y_i \text{ and } x_j = y_j \text{ for all } i < j \leqslant n\,,$$

for all $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{Z}^n$, is a total order compatible with the operation in $\mathbb{Z}^n$.

Note that, from the fact that $\leqslant$ is a total order, every finite subset of $\mathbb{Z}^n$ has a minimum element and a maximum element. For a finite subset A, we will denote the minimum element of A by A_0. From the fact that $\leqslant$ is compatible, we have $(A + x)_0 = A_0 + x$, for all $x \in \mathbb{Z}^n$.

Example 4.1.2. In $\mathbb{Z}^2$, we have $(1, 0) < (0, 1)$ and $(2, 0) < (0, 1)$ (take $i = 2$ in the definition of $\leqslant$) and $(1, 0) < (2, 0)$ (take $i = 1$). In general, $x < y$ in $\mathbb{Z}^2$ if the second component of x is smaller than the second component of y or if x and y have the same second component and the first component of x is smaller than the first component of y. It is now immediate to recognize that the minimum element of the set $A = \{(-1, 1), (0, 1), (1, 1), (1, 0), (2, 0)\}$



is $A_0 = (1, 0)$.

Definition 4.1.3. Let n be a positive integer and Σ be an alphabet. An *n-dimensional language over* Σ is a set of finite and connected Σ-coloured subsets (A, α) of $\mathbb{Z}^n$ with $A_0 = 0$.

An n-dimensional language is *factorial* if for each (A, α) in L and each connected subset B of A, we have that $(B, \alpha|_B) - B_0$ is a member of L.

Recall that, by definition, we always assume that A is a non-empty set in any Σ-coloured subset (A, α) of $\mathbb{Z}^n$ (cf. Definition 3.4.3).

Example 4.1.4. Let $\mathcal{T} = (\mathbb{Z}^n, \tau)$ be an n-dimensional hypercubic tiling with colours in Σ, and consider the set $L(\mathcal{T})$ of all translates $(A, \alpha) - A_0$ with (A, α) a finite, connected Σ-coloured subset of $(\mathbb{Z}^n, \tau)$. We claim that $L(\mathcal{T})$ is an n-dimensional factorial language in the sense of Definition 4.1.3. Clearly, the elements of $L(\mathcal{T})$ are finite and connected Σ-coloured subsets of $\mathbb{Z}^n$. Also, the minimum element of an element $(A, \alpha) - A_0$ in $L(\mathcal{T})$ is $(A - A_0)_0 = A_0 - A_0 = 0$, by the compatibility of $\leqslant$. Moreover, if $(A, \alpha) - A_0$ belongs to $L(\mathcal{T})$ and B is a connected subset of A, then (A, α) is a finite, connected subset of $(\mathbb{Z}^n, \tau)$, and likewise so is $(B, \alpha|_B)$. Thus, by definition, $(B, \alpha|_B) - B_0$ belongs to $L(\mathcal{T})$.

As we will see, the following is the key notion for the generalization we aim for.

Definition 4.1.5. Given an n-dimensional hypercubic tiling $\mathcal{T} = (\mathbb{Z}^n, \tau)$, we define the *language of the tiling* $\mathcal{T}$ or the *tiling language of* $\mathcal{T}$ to be the set

$$L(\mathcal{T}) = \{(A, \alpha) - A_0 \colon (A, \alpha) \text{ is a finite, connected } \Sigma\text{-coloured subset of } (\mathbb{Z}^n, \tau)\} \ .$$

According to Remark 3.4.5, $(A, \alpha) - A_0$ will often be written simply as $A - A_0$.

As usual in any language context, we will refer to the elements of a language, the finite, connected Σ-coloured subsets of $\mathbb{Z}^n$, as *words* and to the indivisible units of a word as *letters*. In the case of the language of a tiling, these are, respectively, the patterns and the tiles, identified up to translation.

Since two patterns in the tiling, equivalent under translation, give rise to the same word in the tiling language, the tiling language of an n-dimensional hypercubic tiling consists of an infinite collection of finite, connected Σ-coloured subsets of $\mathbb{Z}^n$ (but not of $(\mathbb{Z}^n, \tau)$, of course), all of which have minimum element coinciding with the zero element of $\mathbb{Z}^n$ — displayed as if in zero depth layers — whose arrangement of colours occurs somewhere in the tiling. We will see that this is also the efficient notion we desired.

A crucial feature of the language of a one-dimensional tiling is that it is factorial, that is, every word contained in a word from $L(\mathcal{T})$ is again a word in $L(\mathcal{T})$. As we saw in the preceding example, the same happens here. Notice that, to ask for a subset of an element of $L(\mathcal{T})$ to belong to $L(\mathcal{T})$ would not make sense, because the elements of $L(\mathcal{T})$ must have its minimum element placed at zero. In fact, that this is exactly what we unconsciously do with everyday words from every language; but, being so used to reading from left to right (or from right to left, as it may be the case), we forget that every word has a minimum element: its first letter. Even in dimension 1, the origin is there, it is we who miss it.

One-dimensional factorial languages are a particular class of languages, in the language theoretic sense. Therefore, it is reasonable to ask if the same still holds for two-dimensional factorial languages, in the sense of Definition 4.1.3, since two-dimensional languages are established, having been considered and studied in [17]. The answer is no, because, in [17], words are restricted to being rectangular arrays of symbols, whereas the elements of a language in the sense of Definition 4.1.3 are allowed to take any shape, provided it is connected.

4.2 Generalization of the language representation

In this section, we will develop the n-dimensional analogue, for hypercubic tilings, of the inverse semigroup associated with the tiling language, whose identification with the tiling semigroup is instrumental in the theory of one-dimensional tilings. More precisely, we will develop the analogue, for arbitrary dimension, of the inverse semigroup associated with an n-dimensional factorial language and, subsequently, prove that it is isomorphic to the tiling semigroup when the language is the language of a tiling. One could almost say that the difficult part was carried out in the previous section, such is the ease with which this semigroup can now be constructed, allowing us to conclude that the notion adopted for the language of an n-dimensional hypercubic tiling was indeed the appropriate one.

Recall the order $\leqslant$ on $\mathbb{Z}^n$ considered in the previous section. As noticed there, every finite set A has a minimum element, which we denote by A_0, and a maximum element. In particular, the least upper bound $a \vee b$ exists for every $a, b \in \mathbb{Z}^n$. We shall always assume that $\vee$ is prior to the group operation, so that $a \vee b + c$ means $(a \vee b) + c$.

Also, we will always consider the operations of union and intersection between sets to be prior to the operation of set translation, so that, for instance, $A \cup B + x$ means $(A \cup B) + x$, as opposed to $A \cup (B + x)$. We will nevertheless sometimes include the superfluous set of parentheses where it may help to follow the argument.

Let L be an n-dimensional factorial language. Consider the set

$$S(L) = \{(p, P, q)\colon P \in L \text{ and } p, q \in P\} \cup \{0\}$$

and the following operation between the elements of $S(L)$:

$$(p, P, q) * (r, R, s) = (p - q + q \vee r, ((P - q) \cup (R - r)) + q \vee r, s - r + q \vee r)$$

if $(P - q) \cup (R - r) + q \vee r \in L$ and 0 otherwise, for all non-zero elements (p, P, q) and (r, R, s), and all other products equal to 0. Then:

Theorem 4.2.1. *The above binary operation is well-defined and endows $S(L)$ with the structure of a strongly E^*-unitary inverse semigroup with zero.*

Proof. To check that the operation is well-defined, let $(p, P, q), (r, R, s) \in S(L)$ be such that $(p, P, q) * (r, R, s) \neq 0$. Then, by definition,

$$(p, P, q) * (r, R, s) = (p - q + q \vee r, ((P - q) \cup (R - r)) + q \vee r, s - r + q \vee r)$$

with $(P-q)\cup(R-r)+q\vee r$ an element of L. Note that $\big(((P-q)\cup(R-r))+q\vee r\big)_0=0$. In fact, we have

$$\begin{aligned}\min(((P-q)\cup(R-r))+q\vee r) &= \min((P-q+q\vee r)\cup(R-r+q\vee r))\\ &= \min\{\min(P-q+q\vee r),\min(R-r+q\vee r)\}\\ &= \min\{P_0-q+q\vee r, R_0-r+q\vee r\}\\ &= \min\{q\vee r-q, q\vee r-r\}\,,\end{aligned}$$

as $P_0=0$ and $R_0=0$. As $\leqslant$ is a total order on $\mathbb{Z}^n$, we have $q\vee r=q$ or $q\vee r=r$. In the first case, $q\vee r-q=q-q=0$ and $q\vee r-r=q-r\geqslant 0$; in the second, $q\vee r-q=r-q\geqslant 0$ and $q\vee r-r=r-r=0$. Either way, $\min(((P-q)\cup(R-r))+q\vee r)=\min\{q\vee r-q,q\vee r-r\}=0$.

Moreover, the fact that $p\in P$ implies that

$$p-q+q\vee r\in P-q+q\vee r\subseteq((P-q)\cup(R-r))+q\vee r$$

and, similarly, $s-r+q\vee r\in((P-q)\cup(R-r))+q\vee r$. Consequently, we have that $(p-q+q\vee r,((P-q)\cup(R-r))+q\vee r,s-r+q\vee r)\in S(L)$. So $S(L)$ is closed under the operation $*$.

To check that $*$ is associative, let $(p,P,q),(r,R,s),(u,U,v)\in S(L)$. We begin by showing that $\big((p,P,q)*(r,R,s)\big)*(u,U,v)\neq 0$ if and only if $(p,P,q)*\big((r,R,s)*(u,U,v)\big)\neq 0$. Now, $\big((p,P,q)*(r,R,s)\big)*(u,U,v)\neq 0$ if and only if both $((P-q)\cup(Q-r))+q\vee r$ and

$$\begin{aligned}&\Big(\big(((P-q)\cup(R-r))+q\vee r-(s-r+q\vee r)\big)\cup(U-u)\Big)+(s-r+q\vee r)\vee u=\\ &\qquad=\Big(\big(((P-q)\cup(R-r))+r-s\big)\cup(U-u)\Big)+(s-r+q\vee r)\vee u\end{aligned}$$

belong to L, while $(p,P,q)*\big((r,R,s)*(u,U,v)\big)\neq 0$ if and only if both $((R-s)\cup(U-u))+s\vee u$ and

$$\begin{aligned}&\Big((P-q)\cup\big(((R-s)\cup(U-u))+s\vee u-(r-s+s\vee u)\big)\Big)+q\vee(r-s+s\vee u)=\\ &\qquad=\Big((P-q)\cup\big(((R-s)\cup(U-u))+s-r\big)\Big)+q\vee(r-s+s\vee u)\end{aligned}$$

belong to L. Note that the condition

$$X=\Big(\big(((P-q)\cup(R-r))+q\vee r-(s-r+q\vee r)\big)\cup(U-u)\Big)+(s-r+q\vee r)\vee u\in L \quad (4.1)$$

already implies the condition

$$Y=((P-q)\cup(Q-r))+q\vee r\in L\,.$$

In fact, as $Y+(s-r+q\vee r)\vee u$ is a subset of X, if $X\in L$ then

$$Y+(s-r+q\vee r)\vee u-(Y+(s-r+q\vee r)\vee u)_0\in L\,,$$

for L is a factorial language. Since

$$\begin{aligned}(Y+(s-r+q\vee r)\vee u)_0 &= Y_0+(s-r+q\vee r)\vee u\\ &= (((P-q)\cup(Q-r))+q\vee r)_0+(s-r+q\vee r)\vee u\\ &= 0+(s-r+q\vee r)\vee u\\ &= (s-r+q\vee r)\vee u\,,\end{aligned}$$

then $Y+(s-r+q\vee r)\vee u-(Y+(s-r+q\vee r)\vee u)_0=Y$. Therefore, $Y\in L$. Similarly, the condition

$$\Big((P-q)\cup\big(((R-s)\cup(U-u))+s\vee u-(r-s+s\vee u)\big)\Big)+q\vee(r-s+s\vee u)\in L \tag{4.2}$$

implies the condition $((R-s)\cup(U-u))+s\vee u\in L$.

We claim that condition (4.1) is satisfied if and only if condition (4.2) is satisfied. Since $\vee$ is associative and compatible with the operation in $\mathbb{Z}^n$, we have

$$\begin{aligned}r-s+(s-r+q\vee r)\vee u &= (r-s+s-r+q\vee r)\vee(r-s+u)\\ &= (q\vee r)\vee(r-s+u)\\ &= q\vee(r\vee(r-s+u))\\ &= q\vee((r-s+s)\vee(r-s+u))\\ &= q\vee(r-s+s\vee u)\,.\end{aligned}\tag{4.3}$$

Thus, $(s-r+q\vee r)\vee u=s-r+q\vee(r-s+s\vee u)$. Therefore,

$$\begin{aligned}&\Big(\big(((P-q)\cup(R-r))+r-s\big)\cup(U-u)\Big)+(s-r+q\vee r)\vee u=\\ &=\Big(\big(((P-q)\cup(R-r))+r-s\big)\cup(U-u)\Big)+s-r+q\vee(r-s+s\vee u)\\ &=\big((P-q)\cup(R-r)\cup(U-u+s-r)\big)+q\vee(r-s+s\vee u)\\ &=\big((P-q)\cup(R-s+(s-r))\cup(U-u+s-r)\big)+q\vee(r-s+s\vee u)\\ &=\Big((P-q)\cup\big(((R-s)\cup(U-u))+s-r\big)\Big)+q\vee(r-s+s\vee u)\,,\end{aligned}\tag{4.4}$$

and we have our claim. We conclude that $\big((p,P,q)*(r,R,s)\big)*(u,U,v)\neq 0$ if and only if $(p,P,q)*\big((r,R,s)*(u,U,v)\big)\neq 0$.

Additionally, (4.4) also shows that, in case the products $\big((p,P,q)*(r,R,s)\big)*(u,U,v)$ and $(p,P,q)*\big((r,R,s)*(u,U,v)\big)$ are non-zero, their middle coordinates coincide. It remains to show that the same holds true for the first and third coordinates.

By (4.3),

$$p-q+r-s+(s-r+q\vee r)\vee u=p-q+q\vee(r-s+s\vee u) \tag{4.5}$$

and

$$v-u+(s-r+q\vee r)\vee u=v-u+s-r+q\vee(r-s+s\vee u)\,. \tag{4.6}$$

As

$$\begin{aligned}
&\big((p,P,q)*(r,R,s)\big)*(u,U,v) = \\
&= \big(p-q+q\vee r, ((P-q)\cup(R-r))+q\vee r, s-r+q\vee r\big)*(u,U,v) \\
&= \Big(p-q+q\vee r-(s-r+q\vee r)+(s-r+q\vee r)\vee u, \\
&\qquad \Big(\big(((P-q)\cup(R-r))+q\vee r-(s-r+q\vee r)\big)\cup(U-u)\Big)+(s-r+q\vee r)\vee u, \\
&\qquad\quad v-u+(s-r+q\vee r)\vee u\Big) \\
&= \Big(p-q+r-s+(s-r+q\vee r)\vee u, \\
&\qquad \Big(\big(((P-q)\cup(R-r))+r-s\big)\cup(U-u)\Big)+(s-r+q\vee r)\vee u, \\
&\qquad\quad v-u+(s-r+q\vee r)\vee u\Big)
\end{aligned}$$

and

$$\begin{aligned}
&(p,P,q)*\big((r,R,s)*(u,U,v)\big) = \\
&= (p,P,q)*\big(r-s+s\vee u, ((R-s)\cup(U-u))+s\vee u, v-u+s\vee u\big) \\
&= \Big(p-q+q\vee(r-s+s\vee u), \\
&\qquad \Big((P-q)\cup\big(((R-s)\cup(U-u))+s\vee u-(r-s+\vee u)\big)\Big)+q\vee(r-s+s\vee u), \\
&\qquad\quad v-u+s\vee u-(r-s+s\vee u)+q\vee(r-s+s\vee u)\Big) \\
&= \Big(p-q+q\vee(r-s+s\vee u), \\
&\qquad \Big((P-q)\cup\big(((R-s)\cup(U-u))+s-r\big)\Big)+q\vee(r-s+s\vee u), \\
&\qquad\quad v-u+s-r+q\vee(r-s+s\vee u)\Big),
\end{aligned}$$

from equalities (4.4)–(4.6), we conclude that

$$\big((p,P,q)*(r,R,s)\big)*(u,U,v) = (p,P,q)*\big((r,R,s)*(u,U,v)\big).$$

Hence, $S(L)$ is a semigroup (with zero 0, by definition).

It is trivial to check that if $(p,P,q)\in S(L)$, then $(q,P,p)\in S(L)$ and that we have $(p,P,q)*(q,P,p)*(p,P,q) = (p,P,q)$. Thus, $S(L)$ is a regular semigroup. It is also straightforward to check that a non-zero element (p,P,q) of $S(L)$ is an idempotent if and only if $p=q$. To prove that $S(L)$ is an inverse semigroup, we show that the idempotents commute. In fact, for all non-zero idempotents (p,P,p) and (q,Q,q), we have that

$$\begin{aligned}
(p,P,p)*(q,Q,q) &= (p-p+p\vee q, ((P-p)\cup(Q-q))+p\vee q, q-q+p\vee q) \\
&= (p\vee q, ((P-p)\cup(Q-q))+p\vee q, p\vee q) \\
&= (q-q+q\vee p, ((Q-q)\cup(P-p))+q\vee p, p-p+q\vee p) \\
&= (q,Q,q)*(p,P,p)
\end{aligned}$$

if $((P-p)\cup(Q-q))+p\vee q=((Q-q)\cup(P-p))+q\vee p\in L$ and

$$(p,P,p)*(q,Q,q)=0=(q,Q,q)*(p,P,p)$$

otherwise. Thus, $S(L)$ is an inverse semigroup.

Finally, to prove that $S(L)$ is a strongly E^*-unitary inverse semigroup with zero, consider the map $\lambda\colon S(L)\to(\mathbb{Z}^n)^{\overline{0}}$ defined by $0\lambda=\overline{0}$ and $(p,P,q)\lambda=q-p$, for each non-zero element $(p,P,q)\in S(L)$. It is trivial to check that λ, which is 0-restricted by definition, is idempotent-pure, in view of the characterization of the non-zero idempotents mentioned above. That λ is a pre-homomorphism is an easy consequence of the operation in $S(L)$. In fact, if $(p,P,q),(r,R,s)\in S(L)$ are such that $(p,P,q)*(r,R,s)\neq 0$, then

$$\begin{aligned}((p,P,q)*(r,R,s))\lambda &= (p-q+q\vee r,((P-q)\cup(R-r))+q\vee r,s-r+q\vee r)\lambda\\ &= s-r+q\vee r-(p-q+q\vee r)\\ &= s-r+q-p\\ &= (p,P,q)\lambda+(r,R,s)\lambda\,,\end{aligned}$$

as required.

Hence, $S(L)$ is a strongly E^*-unitary inverse semigroup with zero. □

By analogy with the one-dimensional case, we define

Definition 4.2.2. Let L be an n-dimensional factorial language over an alphabet Σ. We call $S(L)$ the *inverse semigroup associated with the n-dimensional factorial language L.*

The next result establishes an isomorphism between the tiling semigroup of an n-dimensional hypercubic tiling and the semigroup from the previous result associated with its language.

Theorem 4.2.3. *Let $\mathcal{T}=(\mathbb{Z}^n,\tau)$ be an n-dimensional hypercubic tiling. The mapping $\phi\colon S(\mathcal{T})\to S(L(\mathcal{T}))$ defined by: $0\phi=0$ and $[a,A,b]\phi=(a-A_0,A-A_0,b-A_0)$, for all non-zero $[a,A,b]\in S(\mathcal{T})$, is an isomorphism.*

Proof. Let $\mathcal{T}=(\mathbb{Z}^n,\tau)$ be an n-dimensional hypercubic tiling and consider its language $L(\mathcal{T})$. By Example 4.1.4, $L(\mathcal{T})$ is an n-dimensional factorial language and so we can consider the semigroup $S(L(\mathcal{T}))$ from Theorem 4.2.1.

If $[a,A,b]\in S(\mathcal{T})$, then $A=(A,\tau|_A)$ is a finite, connected Σ-coloured subset of $(\mathbb{Z}^n,\tau)$ and $a,b\in A$, so that $A-A_0\in L(\mathcal{T})$ with $a-A_0,b-A_0\in A-A_0$. Thus $(a-A_0,A-A_0,b-A_0)$ belongs to $S(L(\mathcal{T}))$. If $[a,A,b]=[c,C,d]$, then $A+x=C$, $a+x=c$, and $b+x=d$, for some translation x of $\mathbb{Z}^n$. Since $C_0=(A+x)_0=A_0+x$, from the compatibility of $\leqslant$, then $x=C_0-A_0$ and so $A+C_0-A_0=C$, $a+C_0-A_0=c$, and $b+C_0-A_0=d$. Therefore $(a-A_0,A-A_0,b-A_0)=(c-C_0,C-C_0,d-C_0)$. Hence, ϕ is well-defined.

That ϕ is bijective is an easy consequence of the definitions of $S(\mathcal{T})$, $S(L(\mathcal{T}))$, and ϕ. In fact, if $[a,A,b]\phi = [c,C,d]\phi$, then $(a-A_0, A-A_0, b-A_0) = (c-C_0, C-C_0, d-C_0)$ by definition of ϕ, so that, taking $x = C_0 - A_0 \in \mathbb{Z}^n$, we have $C = A+x$, $c = a+x$ and $d = b+x$. By definition of $S(\mathcal{T})$, we conclude that $[a,A,b] = [c,C,d]$. Therefore, ϕ is injective. Now let $(p,P,q) \in S(L(\mathcal{T}))$. Then $P = A - A_0$ for some finite, connected Σ-coloured subset A of $(\mathbb{Z}^n, \tau)$, $p = a - A_0$ and $q = b - A_0$ for some $a, b \in A$, by definition of $S(L(\mathcal{T}))$. Therefore, $(p,P,q) = [a,A,b]\phi$, and so ϕ is surjective. Hence, ϕ is a bijection.

To show that ϕ is a homomorphism, we begin by showing that $([a,A,b][c,C,d])\phi \neq 0$ if and only if $[a,A,b]\phi * [c,C,d]\phi \neq 0$, for all $[a,A,b]$ and $[c,C,d]$ non-zero elements of $S(\mathcal{T})$.

So let $[a,A,b], [c,C,d] \in S(\mathcal{T})\backslash\{0\}$. Suppose $([a,A,b][c,C,d])\phi \neq 0$. Since ϕ is 0-restricted, we have $[a,A,b][c,C,d] \neq 0$. Thus, there exist $x, y \in \mathbb{Z}^n$ such that $(A+x) \cup (C+y)$ is a finite and connected Σ-coloured subset of $(\mathbb{Z}^n, \tau)$ and $b + x = c + y$. Then, by definition, $\big((A+x)\cup(C+y)\big) - ((A+x)\cup(C+y))_0 \in L(\mathcal{T})$. Since

$$((A+x)\cup(C+y))_0 = \min((A+x)_0, (C+y)_0) = \min(A_0+x, C_0+y)\,,$$

then

$$\begin{aligned}
&\big((A+x)\cup(C+y)\big) - ((A+x)\cup(C+y))_0 = \\
&\quad = \big((A+x)\cup(C+y)\big) - \min(A_0+x, C_0+y) \\
&\quad = \big(A+x-\min(A_0+x,C_0+y)\big) \cup \big(C+y-\min(A_0+x,C_0+y)\big)
\end{aligned} \tag{4.7}$$

belongs to $L(\mathcal{T})$. Since $b + x = c + y$, we have

$$\begin{aligned}
x - \min(A_0+x, C_0+y) &= -b+c+y+\max(-A_0-x, -C_0-y) \\
&= -b+c+y+\max(-A_0+b-c-y, -C_0-y) \\
&= -b+\max(-A_0+b-c-y+c+y, -C_0-y+c+y) \\
&= -b+\max(b-A_0, c-C_0) \\
&= -b+(b-A_0)\vee(c-C_0)\,.
\end{aligned} \tag{4.8}$$

Similarly,

$$y - \min(A_0+x, C_0+y) = -c + (b-A_0)\vee(c-C_0)\,. \tag{4.9}$$

From (4.7), it follows that

$$\begin{aligned}
&\big((A+x)\cup(C+y)\big) - ((A+x)\cup(C+y))_0 = \\
&\quad = \big(A-b+(b-A_0)\vee(c-C_0)\big) \cup \big(C-c+(b-A_0)\vee(c-C_0)\big) \\
&\quad = \big((A-b)\cup(C-c)\big) + (b-A_0)\vee(c-C_0) \\
&\quad = \big((A-A_0-(b-A_0))\cup(C-C_0-(c-C_0))\big) + (b-A_0)\vee(c-C_0)
\end{aligned} \tag{4.10}$$

belongs to $L(\mathcal{T})$, and so $(a-A_0, A-A_0, b-A_0) * (c-C_0, C-C_0, d-C_0) \neq 0$, that is, $[a,A,b]\phi * [c,C,d]\phi \neq 0$.

Conversely, suppose $[a,A,b]\phi * [c,C,d]\phi \neq 0$. Then, from what we have just seen, $\big((A-b)\cup(C-c)\big)+(b-A_0)\vee(c-C_0) \in L(\mathcal{T})$. By definition of $L(\mathcal{T})$, we have $\big((A-b)\cup(C-c)\big)+(b-A_0)\vee(c-C_0) \in L(\mathcal{T})$ if and only if

$$\begin{aligned}&\big((A-b)\cup(C-c)\big)+(b-A_0)\vee(c-C_0)+z=\\&\qquad=\big(A-b+(b-A_0)\vee(c-C_0)+z\big)\cup\big(C-c+(b-A_0)\vee(c-C_0)+z\big)\end{aligned}$$

is a finite and connected Σ-coloured subset of $(\mathbb{Z}^n,\tau)$, for some $z\in\mathbb{Z}^n$. Since, taking $x=-b+(b-A_0)\vee(c-C_0)+z$ and $y=-c+(b-A_0)\vee(c-C_0)+z$, we have that $x,y\in\mathbb{Z}^n$ are such that $(A+x)\cup(C+y)$ is a finite and connected Σ-coloured subset of $(\mathbb{Z}^n,\tau)$ and

$$\begin{aligned}b+x&=b-b+(b-A_0)\vee(c-C_0)+z\\&=(b-A_0)\vee(c-C_0)+z\\&=c-c+(b-A_0)\vee(c-C_0)+z\\&=c+y\,,\end{aligned}$$

we conclude that $[a,A,b][c,C,d]\neq 0$, and $([a,A,b][c,C,d])\phi\neq 0$, as well.

Therefore, $([a,A,b][c,C,d])\phi\neq 0$ if and only if $[a,A,b]\phi*[c,C,d]\phi\neq 0$. In particular, $([a,A,b][c,C,d])\phi=[a,A,b]\phi*[c,C,d]\phi$ when $[a,A,b][c,C,d]=0$. To conclude, assume that the product $[a,A,b][c,C,d]$ (and, thus, the products $([a,A,b][c,C,d])\phi$ and $[a,A,b]\phi*[c,C,d]\phi$) is non-zero. Then, for $x,y\in\mathbb{Z}^n$ such that $(A+x)\cup(C+y)$ is a finite and connected Σ-coloured subset of $(\mathbb{Z}^n,\tau)$ and $b+x=c+y$, we have

$$\begin{aligned}&([a,A,b][c,C,d])\phi=\\&=[a+x,(A+x)\cup(C+y),d+y]\phi\\&=\big(a+x-((A+x)\cup(C+y))_0,((A+x)\cup(C+y))-((A+x)\cup(C+y))_0,\\&\qquad d+y-((A+x)\cup(C+y))_0\big)\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}[a,A,b]\phi*[c,C,d]\phi&=(a-A_0,A-A_0,b-A_0)*(c-C_0,C-C_0,d-C_0)\\&=(a-b+(b-A_0)\vee(c-C_0),((A-b)\cup(C-c))+(b-A_0)\vee(c-C_0),\\&\qquad d-c+(b-A_0)\vee(c-C_0))\\&=(a-b+(b-A_0)\vee(c-C_0),((A-b)\cup(C-c))+(b-A_0)\vee(c-C_0),\\&\qquad d-c+(b-A_0)\vee(c-C_0))\ .\end{aligned}$$

But then, by (4.8), (4.9) and (4.10), applied to the first, third and second components of $([a,A,b][c,C,d])\phi$ and $[a,A,b]\phi*[c,C,d]\phi$, respectively, we have that these elements coincide. We conclude that ϕ is a homomorphism.

Therefore, $S(\mathcal{T})$ and $S(L(\mathcal{T}))$ are isomorphic. $\square$

Remark 4.2.4. As we mention in Section 3.2, in [32] Lawson showed that the tiling semigroup $S(\mathcal{T})$ of a one-dimensional tiling $\mathcal{T}$, which is identified with a bi-infinite word over a finite alphabet, is isomorphic to the inverse semigroup $S(L)$ associated with a factorial language, consisting of all the words that occur in the bi-infinite word, that is, the patterns of the tiling identified up to translation in $\mathbb{R}$ or, equivalently, in $\mathbb{Z}$, if we think of the tiles as being coloured intervals with length 1. Theorem 4.2.3 shows that exactly the same is true for an n-dimensional hypercubic tiling semigroup. In fact, for $n = 1$ the semigroup in Theorem 4.2.3 amounts to the same as Lawson's representation: both the elements of the semigroups and the operation defined are the same, for if (p, P, q) and (r, R, s) are non-zero elements, then $[(P - q) \cup (R - r)] + q \vee r$ is the outcome of the matching of the out-tile of (p, P, q) with the in-tile of (r, R, s), either by shifting P by $r - q$ in case $q \vee r = r$ or by shifting R by $q - r$ in case $q \vee r = q$ (and the product is non-zero if and only if the matching is a pattern in the tiling), $q \vee r + p - q$ is the position of p, the in-tile of (p, P, q), in $[(P - q) \cup (R - r)] + q \vee r$, and $q \vee r + p - q$ the position of s, the out-tile of (r, R, s). Notice that, here, to be able to decide which of q and r is greater we must have the patterns/words P and R with their minimum element, the leftmost element, aligned, making explicit use of the unconscious procedure we mentioned in the previous section.

Next, we show that the inverse semigroup associated with a factorial language satisfies, just as in the one-dimensional case, the following property:

Proposition 4.2.5. *Let L_1 and L_2 be n-dimensional factorial languages over an alphabet Σ. If $L_1 \subseteq L_2$, then*

$$I = \{(p, P, q) \in S(L_2) \colon P \in L_2 \setminus L_1\} \cup \{0\}$$

is an ideal of $S(L_2)$ and $S(L_1) \simeq S(L_2)/I$.

Proof. By definition, $I \subseteq S_2$. As $I = \{0\}$ trivially implies that $S(L(\mathcal{T}_1)) \simeq S(L(\mathcal{T}_2))/I$, assume that $I \neq \{0\}$. Let $(p, P, q) \in I$ and $(r, R, s) \in S_2$. If $(p, P, q) * (r, R, s) = 0$, then evidently $(p, P, q) * (r, R, s)$ belongs to I; if $(p, P, q) * (r, R, s) \neq 0$, then

$$(p, P, q) * (r, R, s) = (p - q + q \vee r, ((P - q) \cup (R - r)) + q \vee r, s - r + q \vee r)\,,$$

with $((P - q) \cup (R - r)) + q \vee r \in L_2$. In order to obtain a contradiction, suppose that $((P - q) \cup (R - r)) + q \vee r \in L_1$, that is, $(P - q + q \vee r) \cup (R - r + q \vee r) \in L_1$. Since L_1 is factorial and $P - q + q \vee r \subseteq (P - q + q \vee r) \cup (R - r + q \vee r)$, then

$$P - q + q \vee r - (P - q + q \vee r)_0 = P - q + q \vee u - P_0 + q - q \vee r = P$$

(recall that $P_0 = 0$) belongs to L_1, a contradiction since $P \in L_2 \setminus L_1$. A similar argument shows that I is a left ideal as well.

Now, since $L_1 \subseteq L_2$ implies that the non-zero elements of S_1 are precisely the elements in $S_2 \setminus I$ and also because, as we have just seen, $(p, P, q) * (r, R, s) \neq 0$ in S_1 if and only $(p, P, q) * (r, R, s) \notin I$, we conclude that S_1 is isomorphic to the Rees quotient S_2/I. $\square$

To conclude this section, we introduce a notion that with be useful in Chapter 6.

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be n-dimensional hypercubic tilings over an alphabet Σ such that $L(\mathcal{T}_1) = L(\mathcal{T}_2)$. Then, evidently, $S(L(\mathcal{T}_1)) = S(L(\mathcal{T}_2))$ — although $\mathcal{T}_1$ and $\mathcal{T}_2$ need not be the same tiling. In fact, in [25] Kellendonk remarks that two n-dimensional tilings have the same tiling semigroup if and only if each pattern in the first appears as a pattern in the second, and vice-versa. In terms of hypercubic tiling semigroups, this is precisely equivalent to saying that the semigroups are equal if and only if they have the same tiling languages.

In particular, all n-dimensional hypercubic tilings over an alphabet Σ whose language contains every word on Σ have the same associated inverse semigroup — and, thus, isomorphic tiling semigroup. We define:

Definition 4.2.6. Let n be a positive integer and Σ an alphabet. The *free n-dimensional hypercubic tiling semigroup over* Σ, denoted $F(n, \Sigma)$, is the inverse semigroup associated with the n-dimensional factorial language consisting of all finite, connected Σ-coloured subsets of $\mathbb{Z}^n$.

Example 4.2.7. The free one-dimensional hypercubic tiling semigroup over Σ is the semigroup $S(\Sigma^+)$.

In view of Proposition 4.2.5, we immediately have the n-dimensional generalization of Theorem 3.3.2:

Proposition 4.2.8. *Let L be n-dimensional factorial language over an alphabet Σ. Then $S \simeq F(n, \Sigma)/I(L)$, where*

$$I(L) = \{(p, P, q) \in F(n, \Sigma) \colon P \notin L\} \cup \{0\} \ .$$

In the next chapter, we will study another representation of an hypercubic tiling semigroup which relies on the language of the tiling and where one-dimensional tiling semigroups and n-dimensional hypercubic tiling semigroups, with $n \geqslant 2$, display the exactly same behaviour. Next, we further investigate a curious feature of inverse semigroups associated with n-dimensional hypercubic tilings.

4.3 Hypercubic tiling semigroups and generalized Bruck-Reilly extensions

Even though it was then left unremarked, the resemblance between the operations in a Bruck-Reilly extension and in an n-dimensional hypercubic tiling semigroup, as the inverse semigroup associated with the language of the tiling, is unavoidable. In fact, in this representation of a hypercubic tiling semigroup, and in the case of a non-zero product, the outer coordinates multiply exactly as in a Bruck-Reilly extension and the middle coordinates are, in both cases, a combination of the middle coordinates of the elements being multiplied under an action that depends on the third coordinate of the first element and the first coordinate of the second. That was the reason behind the investigations that led to the construction described in Chapter 2. There, we developed an extension that generalizes the Bruck-Reilly extension; here, we will show how hypercubic tiling semigroups fit within this new generalization. More precisely, we will show how every n-dimensional hypercubic tiling semigroup can be produced as a Rees factor semigroup of an inverse subsemigroup of an extension $S(R,T,\theta)$, where all ingredients — not only the group, the semigroup, and the anti-homomorphism, but also the subsemigroup and the ideal — arise most naturally from the tiling.

Fix an n-dimensional hypercubic tiling $\mathcal{T} = (\mathbb{Z}^n, \tau)$ over a finite set Σ. From Theorem 4.2.3, the tiling semigroup $S(\mathcal{T})$ and the semigroup $S(L(\mathcal{T}))$, associated with the language of the tiling, are isomorphic. We will describe $S(L(\mathcal{T}))$ as a Rees factor semigroup of an inverse subsemigroup of a certain extension $S(R,T,\theta)$.

In Lemma 4.1.1, we adopted a compatible total order on $\mathbb{Z}^n$. Under this order, $\mathbb{Z}^n$ is, thus, a lattice ordered group. By Lemma 2.1.7, the positive cone of $\mathbb{Z}^n$ is a cancellative monoid without non-trivial units whose principal left ideals form a semilattice under intersection. Throughout this section, let R denote the positive cone of $\mathbb{Z}^n$ under the order from Lemma 4.1.1. By Lemma 2.1.3 (ii), the least upper bound $p \vee q$ of $p, q \in R$ always exists in R and, by definition, $p * q$ is the unique element, in R, such that $(p*q)q = p \vee q$. Note that, in this monoid, $p \vee q = max\,(p,q)$, and so $(p*q)q = max\,(p,q)$, that is, $p*q = max\,(p,q) - q$.

Example 4.3.1. For $n = 2$, we have

$$x = (x_1, x_2) \geqslant (0,0) \Leftrightarrow x_1 = x_2 = 0 \text{ or } (x_1 > 0 \text{ and } x_2 = 0) \text{ or } x_2 > 0\,,$$

and so

$$\begin{aligned} R &= \big\{(x_1, 0) \in \mathbb{Z}^2 \colon x_1 \geqslant 0\big\} \cup \big\{(x_1, x_2) \in \mathbb{Z}^2 \colon x_2 \geqslant 1\big\} \\ &= \big(\mathbb{Z}_0^+ \times \{0\}\big) \cup \big(\mathbb{Z} \times \mathbb{Z}^+\big) \ : \end{aligned}$$

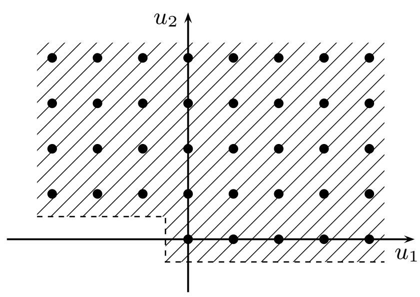
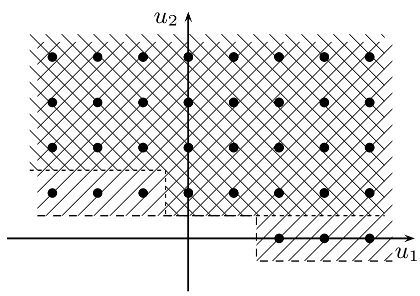


Figure 4.1: The positive cone of $\mathbb{Z}^2$

Thus, for instance $(R + (2,0)) \cap (R + (0,1)) = R + (0,1)$, in accordance with Lemma 2.1.3 (ii), as $(2,0) < (0,1)$ implies that $(2,0) \vee (0,1) = (0,1)$. Therefore, we have

$(2,0) * (0,1) = (0,1) - (0,1) = (0,0)$ and $(0,1) * (2,0) = (0,1) - (2,0) = (-2,1)$.



Recall from Remark 3.4.4 that, if (A,α) and (B,β) are Σ-coloured subsets of $\mathbb{Z}^n$ and $\alpha|_{A\cap B} = \beta|_{A\cap B}$, then $\alpha \cup \beta \colon A \cup B \to \Sigma$ denotes the (well-defined) map

$$(\alpha \cup \beta)(x) = \begin{cases} \alpha(x) & \text{if } x \in A \\ \beta(x) & \text{if } x \in B\,. \end{cases}$$

In particular, $(A \cup B, \alpha \cup \beta)$ is, in this case, a Σ-coloured subset of $\mathbb{Z}^n$.

Let T be the set of all finite Σ-coloured subsets (A,α) of $\mathbb{Z}^n$ together with an element $\tilde{0}$. (Notice that the Σ-coloured subsets of $\mathbb{Z}^n$ that consititute the non-zero elements of T need not be connected.) Extending the partial binary operation between Σ-coloured subsets of $\mathbb{Z}^n$ in the previous paragraph, we will be able to endow T with the structure of a semilattice:

Lemma 4.3.2. *The set T of all finite Σ-coloured subsets of $\mathbb{Z}^n$ together with $\tilde{0}$, equipped with the operation defined by*

$$(A,\alpha)\cdot(B,\beta) = \begin{cases} (A \cup B, \alpha \cup \beta) & \textit{if } \alpha|_{A\cap B} = \beta|_{A\cap B} \\ \tilde{0} & \textit{otherwise} \end{cases}$$

and $(A,\alpha)\cdot\tilde{0} = \tilde{0} = \tilde{0}\cdot(A,\alpha)$, for all $(A,\alpha),(B,\beta) \in T$, constitutes a semilattice (with zero).

Proof. In Section 3.4, we observed that the following rule defines a partial order on the set of Σ-coloured subsets of $\mathbb{Z}^n$: for all Σ-coloured subsets (A,α) and (B,β) of $\mathbb{Z}^n$, we defined $(A,\alpha) \leqslant (B,\beta)$ if and only if $A \subseteq B$ and $\beta|_A = \alpha$. So, in particular, we have a partial order on the set of finite Σ-coloured subsets of $\mathbb{Z}^n$. Thus, setting $\tilde{0} \leqslant \tilde{0}$ and $(A,\alpha) \leqslant \tilde{0}$ for each finite Σ-coloured subset of $\mathbb{Z}^n$, we have a partial order on T.

We claim that T is a sup-semilattice under this partial order. To see this, let (A,α) and (B,β) be finite Σ-coloured subsets of $\mathbb{Z}^n$. Notice that, if there exists a finite Σ-coloured subset (C,γ) of $\mathbb{Z}^n$ such that $(A,\alpha),(B,\beta) \leqslant (C,\gamma)$, then $\alpha|_{A\cap B} = \beta|_{A\cap B}$, whence $(A \cup B, \alpha \cup \beta)$ is a well-defined Σ-coloured subset of $\mathbb{Z}^n$. Further, it is trivial to check that, in this case, $(A,\alpha) \vee (B,\beta) = (A \cup B, \alpha \cup \beta)$. Of course, if there exists no such element (C,γ), then $(A,\alpha) \vee (B,\beta) = \tilde{0}$ by construction. Therefore, T forms a sup-semilattice under the partial order from Section 3.4. In particular, the binary operation $\vee$ is associative, commutative and idempotent. Since this operation coincides with the one defined on T, we conclude that T is a semilattice with zero. $\square$

Remark 4.3.3. Notice that the natural partial order on T is the reverse of the order from Section 3.4, namely

$$(A, \alpha) \leqslant^r (B, \beta) \Leftrightarrow B \subseteq A \text{ and } \alpha|_B = \beta\,.$$

The remaining building block to have an extension $S(R, T, \theta)$ is an anti-homomorphism $\theta\colon R \to \operatorname{End} T$. Its construction takes on the following two lemmas.

Lemma 4.3.4. *For $p \in R$, the map $\theta_p\colon T \to T$ defined by $\tilde{0}\theta_p = \tilde{0}$ and $(A, \alpha)\theta_p = (A, \alpha) + p$, the Σ-coloured subset $(A+p, \alpha+p)$ of $\mathbb{Z}^n$ with $\alpha+p\colon A+p \to \Sigma$ defined by $(\alpha+p)(x) = \alpha(x-p)$, for all $x \in A + p$, is a homomorphism.*

Proof. Fix $p \in R$. The considerations preceding Remark 3.4.4 allow us to conclude that θ_p is well-defined. By definition, we have

$$(A, \alpha)\theta_p \cdot \tilde{0}\theta_p = ((A, \alpha) + p) \cdot \tilde{0} = \tilde{0} = \tilde{0}\theta_p = ((A, \alpha) \cdot \tilde{0})\theta_p\,,$$

so that $((A, \alpha) \cdot \tilde{0})\theta_p = (A, \alpha)\theta_p \cdot \tilde{0}\theta_p$. Similarly, $(\tilde{0} \cdot (A, \alpha))\theta_p = \tilde{0}\theta_p \cdot (A, \alpha)\theta_p$. Now let $(A, \alpha), (B, \beta) \in T \setminus \{\tilde{0}\}$.

Notice that $\alpha|_{A\cap B} = \beta|_{A\cap B}$ if and only if $(\alpha + p)|_{(A+p)\cap(B+p)} = (\beta + p)|_{(A+p)\cap(B+p)}$. In fact, assume that $\alpha|_{A\cap B} = \beta|_{A\cap B}$ and let $y \in (A + p) \cap (B + p)$. Then $y - p \in A \cap B$ and

$$(\alpha + p)(y) = \alpha(y - p) = \beta(y - p) = (\beta + p)(y)\,.$$

Conversely, assume that $(\alpha + p)|_{(A+p)\cap(B+p)} = (\beta + p)|_{(A+p)\cap(B+p)}$ and let $x \in A \cap B$. Then $x + p \in (A + p) \cap (B + p)$ and

$$\alpha(x) = (\alpha + p)(x + p) = (\beta + p)(x + p) = \beta(x)\,.$$

Consequently,

$$\begin{aligned}
(A, \alpha)\theta_p \leqslant (B, \beta)\theta_p &\Leftrightarrow (A, \alpha) + p \leqslant (B, \beta) + p \\
&\Leftrightarrow (A + p, \alpha + p) \leqslant (B + p, \beta + p) \\
&\Leftrightarrow B + p \subseteq A + p \text{ and } (\alpha + p)|_{(B+p)} = \beta + p \\
&\Leftrightarrow B \subseteq A \text{ and } \alpha|_B = \beta \\
&\Leftrightarrow (A, \alpha) \leqslant (B, \beta)\,.
\end{aligned}$$

Since T is a semilattice, this implies that θ_p is in fact an order automorphism of T. □

Lemma 4.3.5. *Consider the map $\theta\colon R \to \operatorname{End} T$ defined by $p\theta = \theta_p$, for all $p \in R$. Then θ is an anti-homomorphism.*

Proof. Condition (AH1) of anti-homomorphism (cf. Section 2.3) was proved in the previous lemma. As for condition (AH2), it is equivalent to proving that $(u\theta_p)\theta_q = u\theta_{p+q}$, for all $u \in T$

and $p, q \in R$, since the operation of $\mathbb{Z}^n$ is commutative. In fact, we have $(\tilde{0}\theta_p)\theta_q = \tilde{0} = \tilde{0}\theta_{p+q}$, trivially, and

$$\begin{aligned}((A,\alpha)\theta_p)\theta_q &= ((A,\alpha)+p)\theta_q = (A+p,\alpha+p)\theta_q = (A+p,\alpha+p)+q \\ &= ((A+p)+q,(\alpha+p)+q) = (A+(p+q),\alpha+(p+q)) = (A,\alpha)\theta_{p+q}\,,\end{aligned}$$

almost as trivially, since the maps $(\alpha + p) + q$ and $\alpha + (p + q)$ coincide. In fact, for all $x \in (A + p) + q = A + (p + q)$,

$$\begin{aligned}((\alpha+p)+q)(x) &= (\alpha+p)(x-q) = \alpha((x-q)-p) = \alpha(x-q-p) \\ &= \alpha((x-(q+p)) = \alpha(x-(p+q)) = (\alpha+(p+q))(x)\,,\end{aligned}$$

according to Remark 3.4.5. Finally, condition (AH3) is satisfied, as θ_0 is the identity map on T: by definition, $0\theta_0 = 0$ and $(A, \alpha)\theta_0 = (A, \alpha) + 0 = (A, \alpha)$, for all $(A, \alpha) \in T \setminus \{\tilde{0}\}$. □

Thus,

Proposition 4.3.6. *Let R be the positive cone of $\mathbb{Z}^n$, let T be the set of all finite Σ-coloured subsets of $\mathbb{Z}^n$ together with $\tilde{0}$, equipped with the operation defined on the non-zero elements by*

$$(A,\alpha)\cdot(B,\beta) = \begin{cases}(A\cup B,\alpha\cup\beta) & \text{if } \alpha|_{A\cap B} = \beta|_{A\cap B} \\ \tilde{0} & \text{otherwise}\end{cases}$$

and let $\theta\colon R \to \operatorname{End} T$ be the map defined by

$$\tilde{0}(p\theta) = \tilde{0}\theta_p = \tilde{0} \quad \text{and} \quad (A,\alpha)(p\theta) = (A,\alpha)\theta_p = (A,\alpha)+p\,,$$

for all $p \in R$ and all $(A, \alpha) \in T$. Then, $S(R, T, \theta) = R \times T \times R$, equipped with the operation defined by:

$$(p,u,q)(r,v,s) = (p-q+q\vee r, u\theta_{q\vee r-q}\cdot v\theta_{q\vee r-r}, s-r+q\vee r)\,,$$

for all $(p, u, q), (r, v, s) \in R \times T \times R$, is a E-unitary inverse semigroup.

Proof. Since the order considered on $\mathbb{Z}^n$ is a compatible total order by Lemma 4.1.1, we have that, under this order, $\mathbb{Z}^n$ is a lattice ordered group. Thus, by Lemma 2.1.7, the positive cone R of $\mathbb{Z}^n$ is a cancellative monoid without non-trivial units whose principal left ideals form a semilattice under intersection. From Lemma 4.3.2, T, being a semilattice, is in particular a semigroup, and, by Lemma 4.3.5, $\theta\colon R \to \operatorname{End} T$ is an anti-homomorphism. Therefore, by Theorem 2.3.1, $S(R, T, \theta)$ is a semigroup with respect to the operation defined by: for all $(p, u, q), (r, v, s) \in R \times T \times R$,

$$\begin{aligned}(p,u,q)(r,v,s) &= ((r*q)p, u\theta_{r*q}\cdot v\theta_{q*r}, (q*r)s) \\ &= (r\vee q-q+p, u\theta_{r\vee q-q}\cdot v\theta_{q\vee r-r}, q\vee r-r+s) \\ &= (p-q+q\vee r, u\theta_{q\vee r-q}\cdot v\theta_{q\vee r-r}, s-r+q\vee r)\,,\end{aligned}$$

from the definition of $*$ (cf. Section 2.1) and the fact that the operation on $\mathbb{Z}^n$ is commutative. Finally, from Proposition 2.3.8, the fact that T is a semilattice implies that $S(R,T,\theta)$ is an E-unitary inverse semigroup. $\square$

Remark 4.3.7. 1. Because T has a zero element,

$$\begin{aligned} S(R,T,\theta) &= \{(p,u,q)\colon p,q\in R \text{ and } u\in T\} \\ &= \left\{(p,(A,\alpha),q)\colon p,q\in R \text{ and } (A,\alpha)\in T\setminus\{\tilde{0}\}\right\}\cup\left\{(p,\tilde{0},q)\colon p,q\in R\right\}\,, \end{aligned}$$

where $W=\left\{(p,\tilde{0},q)\colon p,q\in R\right\}$ is an ideal of $S(R,T,\theta)$. In fact, since

$$W=\left\{(p,u,q)\in S(R,T,\theta)\colon u\in\{\tilde{0}\}\right\}\,,$$

where $\{\tilde{0}\}$ is a two-sided ideal of T such that, for all $r\in R$, the restriction $\theta_p|_{\{\tilde{0}\}}$ is an endomorphism of $\{\tilde{0}\}$, by Lemma 2.3.11 we conclude that W is a two-sided ideal of $S(R,T,\theta)$. (Equivalently, we could have shown, just as easily, that $(p,u,q)(r,\tilde{0},s)$ and $(r,\tilde{0},s)(p,u,q)$ belong to W for all $(p,u,q)\in S(R,T,\theta)$ and $(r,\tilde{0},s)\in W$.)

To simplify matters, we will henceforth identify $S(R,T,\theta)$ with $S(R,T,\theta)/W$, which is the same as considering $S(R,T,\theta)=(R\times(T\setminus\{\tilde{0}\})\times R)\cup\{0\}$.

2. Under this identification, given two non-zero elements $(p,(A,\alpha),q),(r,(B,\beta),s)$ in $S(R,T,\theta)$, we have $(p,(A,\alpha),q)(r,(B,\beta),s)\neq 0$ if and only if $(p,(A,\alpha),q)(r,(B,\beta),s)$ belongs to $S(R,T,\theta)\setminus W$, that is, $(A,\alpha)\theta_{q\vee r-q}\cdot(B,\beta)\theta_{q\vee r-r}\neq\tilde{0}$ in T. By definition of θ, this is the case if and only if

$$((A,\alpha)+q\vee r-q)\cdot((B,\beta)+q\vee r-r)\neq\tilde{0}$$

in T; by definition of the operation on T, we must have

$$(\alpha-q+q\vee r)|_{(A-q+q\vee r)\cap(B-r+q\vee r)}=(\beta-r+q\vee r)|_{(A-q+q\vee r)\cap(B-r+q\vee r)}\,,$$

where we have used the fact that the operation on $\mathbb{Z}^n$ is commutative.

3. Finally, in case $(p,(A,\alpha),q)(r,(B,\beta),s)\neq 0$, we have

$$(p,(A,\alpha),q)(r,(B,\beta),s)=(p-q+q\vee r,(C,\gamma),s-r+q\vee r)$$

with $C=(A-q+q\vee r)\cup(B-r+q\vee r)$ and

$$\gamma=(\alpha-q+q\vee r)\cup(\beta-r+q\vee r)\colon(A-q+q\vee r)\cup(B-r+q\vee r)\to\Sigma$$

is defined by

$$\begin{aligned}\gamma(x)&=\begin{cases}(\alpha-q+q\vee r)(x) & \text{if } x\in A-q+q\vee r\\ (\beta-r+q\vee r)(x) & \text{if } x\in B-r+q\vee r\end{cases}\\ &=\begin{cases}\alpha(x+q-q\vee r) & \text{if } x\in A-q+q\vee r\\ \beta(x+r-q\vee r) & \text{if } x\in B-r+q\vee r\,,\end{cases}\end{aligned}$$

for all $x\in(A-q+q\vee r)\cup(B-r+q\vee r)$.

Our aim is to find a suitable inverse subsemigroup S of $S(R,T,\theta)$ and an ideal K of S such that S/K is isomorphic to $S(L(\mathcal{T}))$. Bearing in mind that the Σ-coloured subset P underlying a non-zero element (p,P,q) of $S(L(\mathcal{T}))$ is connected and has minimum element placed at zero and also that $p,q \in P$, consider the subset of $S(R,T,\theta)$

$$S = \big\{(p,(A,\alpha),q) \in R \times (T \setminus \{\tilde{0}\}) \times R \colon 0,p,q \in A, A \subseteq R \text{ and } A \text{ is connected } \big\} \cup \{0\} \ .$$

Proposition 4.3.8. *The subset S is an inverse subsemigroup of $S(R,T,\theta)$.*

Proof. Since $S \setminus \{0\} \subseteq R \times (T \setminus \{\tilde{0}\}) \times R$, we can consider the identification from the previous remark. And, in view of the multiplication in $S(R,T,\theta)$ under this identification, the proof is straightforward checking. In fact, for all $(p,(A,\alpha),q),(r,(B,\beta),s) \in S$ whose product is non-zero (cf. Remark 4.3.7 (3)), we have:

- as $\leqslant$ is a total order in $\mathbb{Z}^n$, either $q \vee r = q$ or $q \vee r = r$, and so there are two possibilities:
$$(A - q + q \vee r) \cup (B - r + q \vee r) = A \cup (B - r + q \vee r)$$
or
$$(A - q + q \vee r) \cup (B - r + q \vee r) = (A - q + q \vee r) \cup B\,,$$
respectively, so that 0 belongs to $(A - q + q \vee r) \cup (B - r + q \vee r)$ as $0 \in A \cap B$;
- since $p \in A$ by assumption, then $p - q + q \vee r$ belongs to $A - q + q \vee r$, and hence to $(A - q + q \vee r) \cup (B - r + q \vee r)$;
- similarly, $s - r + q \vee r$ belongs to $(A - q + q \vee r) \cup (B - r + q \vee r)$;
- since $q \vee r \geqslant q$ and $q \vee r \geqslant r$ imply that $q \vee r - q \geqslant 0$ and $q \vee r - r \geqslant 0$, respectively, we have $(A - q + q \vee r) \cup (B - r + q \vee r) \subseteq R$, as $A, B \subseteq R$ by definition of S;
- finally, since A and B are connected by definition of S, we have that $A - q + q \vee r$ and $B - r + q \vee r$ are connected sets. Moreover, since $q \in A$, then $q \vee r = q - q + q \vee r$ belongs to $A - q + q \vee r$ and, similarly, $q \vee r = r - r + q \vee r$ is in $B - r + q \vee r$, so that $q \vee r \in (A - q + q \vee r) \cap (B - r + q \vee r)$. Thus, $(A - q + q \vee r) \cap (B - r + q \vee r) \neq \emptyset$, and so $(A - q + q \vee r) \cup (B - r + q \vee r)$ is a connected set.

Therefore, $(p,(A,\alpha),q)(r,(B,\beta),s) \in S$, and, hence, S is a semigroup. By Proposition 2.3.8 (2) and the fact that T is a semilattice, it is immediate that S is an inverse subsemigroup of $S(R,T,\theta)$, since the inverse of a non-zero element $(p,(A,\alpha),q)$ is the element $(q,(A,\alpha),p)$, which belongs to S. ∎

Now to account for the fact that the colouring map underlying a non-zero element of $S(L(\mathcal{T}))$ is a restriction of the map $\tau\colon \mathbb{Z}^n \to \Sigma$ with which we identify the tiling $\mathcal{T}$, consider

$$\begin{aligned} J = \{(p,(A,\alpha),q) \in S(R,T,\theta)\colon & \\ & (A,\alpha) + x \text{ is not a coloured subset of } (\mathbb{Z}^n,\tau) \text{ for any } x \in \mathbb{Z}^n\} \cup \{0\} \ . \end{aligned}$$

Proposition 4.3.9. *The set J is an ideal of $S(R,T,\theta)$.*

Proof. In order to take advantage of Lemma 2.3.11, we begin by showing that J is a ideal of $S(R,T,\theta)$ in its original form. By this result, it suffices to show that

$$I = \{(A,\alpha) \in T\colon (A,\alpha)+x \text{ is not a coloured subset of } (\mathbb{Z}^n,\tau) \text{ for any } x \in \mathbb{Z}^n\} \cup \{\tilde{0}\}$$

is an ideal of T with the property that θ_r restricts to an endomorphism of I, for all $r \in R$.

To show that I is an ideal of T, let $(A,\alpha) \in I$ and $u \in T$. If $u = \tilde{0}$ or $u = (B,\beta)$ is such that $(A,\alpha)\cdot(B,\beta) = \tilde{0}$ (which implies that $(B,\beta)\cdot(A,\alpha) = \tilde{0}$, as we saw in the proof of Lemma 4.3.2), then $(A,\alpha)\cdot u = u \cdot (A,\alpha) = \tilde{0}$, and so trivially $(A,\alpha)\cdot u, u\cdot(A,\alpha) \in I$; if $u = (B,\beta)$ is such that $(A,\alpha)\cdot(B,\beta) \neq \tilde{0}$, then $\alpha|_{A\cap B} = \beta|_{A\cap B}$ and $(A,\alpha)\cdot(B,\beta) = (A\cup B, \alpha\cup\beta)$. In order to obtain a contradiction, suppose that $(A\cup B,\alpha\cup\beta) \notin I$. Then $(A\cup B,\alpha\cup\beta)+x \leqslant (\mathbb{Z}^n,\tau)$, for some $x \in \mathbb{Z}^n$, that is,

$$(A\cup B,\alpha\cup\beta)+x = ((A\cup B)+x,(\alpha\cup\beta)+x) = ((A+x)\cup(B+x),(\alpha+x)\cup(\beta+x))$$

is a Σ-coloured subset of $(\mathbb{Z}^n,\tau)$. But then, in particular, $(A,\alpha)+x = (A+x,\alpha+x)$ is a Σ-coloured subset of $(\mathbb{Z}^n,\tau)$, a contradiction since $(A,\alpha) \in I$.

Now, let $r \in R$ and $u \in I$. If $u = 0$, then $0\theta_r = 0 \in I$, so let $u = (A,\alpha)$. Again, suppose that $(A,\alpha)\theta_r = (A,\alpha)+r \notin I$ in order to obtain a contradiction. Then $((A,\alpha)+r)+x$ is a coloured subset of $(\mathbb{Z}^n,\tau)$, for some $x \in \mathbb{Z}^n$, or, equivalently, $(A,\alpha)+(r+x)$ is a coloured subset of $(\mathbb{Z}^n,\tau)$, with $r+x \in \mathbb{Z}^n$. But then $(A,\alpha) \notin I$, a contradiction. Therefore, $(A,\alpha)\theta_r \in I$.

By Lemma 2.3.11, we have that J is an ideal of $S(R,T,\theta)$.

Finally, since $J \setminus \{0\} \subseteq R \times (T \setminus \{\tilde{0}\}) \times R$, we conclude that J is an ideal of $S(R,T,\theta)$, under the identification between $S(R,T,\theta)$ and $S(R,T,\theta)/W$ made in Remark 4.3.7. ☐

From Propositions 4.3.8 and 4.3.9, $S \cap J$ is an ideal of $S(R,T,\theta)$.

We can now show the main result of this section:

Theorem 4.3.10. *Let $\mathcal{T} = (\mathbb{Z}^n,\tau)$ be an n-dimensional hypercubic tiling over a finite set Σ. Consider the lattice ordered group $\mathbb{Z}^n$, its positive cone R, the set T of all finite Σ-coloured subsets (A,α) of $\mathbb{Z}^n$ together with $\tilde{0}$, equipped with the operation*

$$(A,\alpha)\cdot(B,\beta) = \begin{cases} (A\cup B,\alpha\cup\beta) & \text{if } \alpha|_{A\cap B} = \beta|_{A\cap B} \text{ exists} \\ \tilde{0} & \text{otherwise} \end{cases}$$

and the anti-homomorphism $\theta\colon R \to \operatorname{End} T$ defined by $\tilde{0}\theta_p = \tilde{0}$ and $(A,\alpha)\theta_p = (A,\alpha)+p$, for all $p \in R$ and $(A,\alpha) \in T$. Then $S(\mathcal{T})$ divides an inverse subsemigroup of $S(R,T,\theta)$. More precisely, if

$$S = \{(p,(A,\alpha),q) \in S(R,T,\theta)\colon 0,p,q \in A, A \subseteq R \text{ and } A \text{ is connected }\} \cup \{0\}$$

and

$$J = \{(p,(A,\alpha),q) \in S(R,T,\theta)\colon \\ (A,\alpha)+x \textit{ is not a coloured subset of } (\mathbb{Z}^n,\tau) \textit{ for any } x \in \mathbb{Z}^n\} \cup \{0\}\ ,$$

then $S(\mathcal{T}) \simeq S/(S\cap J)$.

Proof. Since, by Theorem 4.2.3, the tiling semigroup $S(\mathcal{T})$ and the semigroup $S(L(\mathcal{T}))$, associated with the language of the tiling, are isomorphic, it suffices to prove that $S(L(\mathcal{T})) \simeq S/(S\cap J)$. In fact, we will prove that $S(L(\mathcal{T})) = S/(S\cap J)$.

The non-zero elements of $S/(S\cap J)$ are the members of $S\setminus(S\cap J) = S\setminus J$, that is, the triples $(p,(A,\alpha),q)$ from $R\times(T\setminus\{0\})\times R$ with $A\subseteq R$, $0,p,q\in A$, A connected (and finite, by definition of T) such that $(A,\alpha)+x \leqslant (\mathbb{Z}^n,\tau)$, for some $x\in\mathbb{Z}^n$. Since the conditions $A\subseteq R$ and $0\in A$ imply that $A_0=0$, we conclude that the non-zero elements of $S/(S\cap J)$ have middle coordinate the translate of a finite and connected Σ-coloured subset of $(\mathbb{Z}^n,\tau)$, the translation being such that the minimum element of A is mapped onto 0. Also, in both semigroups, the first and third coordinates of the non-zero elements are vertices belonging to the set from the second coordinate. Thus, $S(L(\mathcal{T}))$ and $S/(S\cap J)$ have the same non-zero elements.

Regarding the product, we have that $(p,(A,\alpha),q)(r,(B,\beta),s)$ is non-zero in $S/(S\cap J)$ if and only if $(p,(A,\alpha),q)(r,(B,\beta),s)\notin J$, or, equivalently, $(p,(A,\alpha),q)(r,(B,\beta),s)\neq 0$ in S and, in view of Remark 4.3.7 (1), $\big((A-q+q\vee r)\cup(B-r+q\vee r),(\alpha-q+q\vee r)\cup(\beta-r+q\vee r)\big)+x$ is a coloured subset of $(\mathbb{Z}^n,\tau)$ for some $x\in\mathbb{Z}^n$. Now by Remark 4.3.7 (2), we have that $(p,(A,\alpha),q)(r,(B,\beta),s)\neq 0$ in S if and only if

$$(\alpha-q+q\vee r)|_{(A-q+q\vee r)\cap(B-r+q\vee r)} = (\beta-r+q\vee r)|_{(A-q+q\vee r)\cap(B-r+q\vee r)}\,.$$

On the other hand, the product $(p,P,q)(r,Q,s)$ is non-zero in $S(L(\mathcal{T}))$ if and only if $(P-q+q\vee r)\cup(R-r+q\vee r)$ belongs to $L(\mathcal{T})$, that is, is the translate of some finite, connected Σ-coloured subset of $(\mathbb{Z}^n,\tau)$. Recall that, in view of the abuse of notation adopted in Remark 3.4.5, to have $(P-q+q\vee r)\cup(R-r+q\vee r)\in L(\mathcal{T})$ implies that the colouring maps agree on the overlap, which is exactly what the condition above on α and β says. Therefore, a product is non-zero in $S/(S\cap J)$ if and only if it is non-zero in $S(L(\mathcal{T}))$.

Finally, we check that the multiplication in $S(L(\mathcal{T}))$ and in $S/(S\cap J)$ is the same, which was in fact our original motivation. Since we already know that the product is non-zero in $S/(S\cap J)$ if and only if it is non-zero in $S(L(\mathcal{T}))$, it remains to show that such products yield the same result in both semigroups, which is trivial in view of the definition of $S(L(\mathcal{T}))$ and Remark 4.3.7 (3).

We hence conclude that $S(L(\mathcal{T}))\simeq S/(S\cap J)$. □

In particular, we have the following:

Corollary 4.3.11. *Let R be the positive cone of the lattice ordered group $\mathbb{Z}^n$. Then any n-dimensional hypercubic tiling semigroup divides an extension of the bisimple inverse semigroup $R^{-1} \circ R$.*

Proof. Recall from Section 2.2 that, under the conditions satisfied by the monoid R, the pair (R, R) is an RP-system. We can therefore consider the bisimple inverse semigroup $R^{-1} \circ R$ from Theorem 2.2.4.

Thus, on the one hand $S(\mathcal{T})$ is a homomorphic image of a subsemigroup of $S(R, T, \theta)$, by the previous theorem; that is, $S(\mathcal{T})$ divides $S(R, T, \theta)$. On the other hand, $R^{-1} \circ R$ is a homomorphic image of $S(R, T, \theta)$, by Proposition 2.3.6, or, equivalently, $S(R, T, \theta)$ is an extension of $R^{-1} \circ R$.

$$\begin{array}{ccc} S & \hookrightarrow & S(R,T,\theta) \\ \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow \\ S(\mathcal{T}) \simeq S/(S \cap J) & & R^{-1} \circ R \end{array}$$

$\square$

Chapter 5

P*-semigroup representation

As shown in [41], every tiling semigroup is strongly E^*-unitary, and, by Theorem 4.2.1, so is the inverse semigroup associated with an n-dimensional factorial language. A natural question to ask, thus, is how can these semigroups be represented as P^*-semigroups. As we recalled in Chapter 1, a possible answer comes as the meeting point of several different contributions: the constructive proof of the P-Theorem given by Munn [47], the theory regarding E-unitary covers of inverse semigroups due to McAlister and Reilly [39], and McAlister's proof of the connection between strongly E^*-unitary inverse semigroups and Rees quotients of E-unitary inverse semigroups [42]. We refer the reader to Section 1.1 of Chapter 1 for details.

In this chapter, we will deal with the more general case of inverse semigroups associated with n-dimensional factorial languages, rather than with n-dimensional hypercubic tiling semigroups, since this does not require any additional (and possibly unjustified) effort. Since the description found depends entirely on the language, we immediately obtain a representation as a P^*-semigroup of an n-dimensional hypercubic tiling semigroup when the language is the language of the tiling, emphasizing how much information about the tiling and the tiling semigroup is encoded in the tiling language.

As motivation, we will implement the construction of the representation as a P^*-semigroup of the inverse semigroup associated with a (one-dimensional) factorial language as described in Section 5.1. In Section 5.2, we will give a direct proof of the generalization to n-dimensional factorial languages of the representation obtained in the first section. Finally, in Section 5.3, we provide some examples that indicate the usefulness of the construction, and show some of its subtleties too.

5.1 The construction for one-dimensional factorial languages

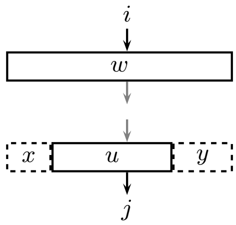
Let L be a (one-dimensional) factorial language over a finite alphabet Σ and let $S(L)$ be the inverse semigroup associated with L. In what follows, a different representation — from the one given in the definition of $S(L)$ (cf. the subsection of Section 3.2 concerning the inverse semigroup associated with a (one-dimensional) factorial language) — of the non-zero elements of $S(L)$ will be more convenient. Since each non-zero element of $S(L)$ is characterized by its underlying word, the in-letter, and the out-letter, given a non-zero element of $S(L)$ with underlying word w, we may uniquely represent it as a triple (i, w, j), where i, respectively, j, stands for the position of the in-letter, respectively, the out-letter, of the element. Therefore,

$$S(L) = \{(i, w, j) \in \mathbb{Z} \times L \times \mathbb{Z} \colon 1 \leqslant i \leqslant |w| \text{ and } 1 \leqslant j \leqslant |w|\} \cup \{0\}$$

and, in particular,

$$E_{S(L)} = \{(i, w, i) \in \mathbb{Z} \times L \times \mathbb{Z} \colon 1 \leqslant i \leqslant |w|\} \cup \{0\} \ .$$

In [58], it is shown that, in an arbitrary tiling semigroup $S(\mathcal{T})$, we have $[a, A, b] \leqslant [c, C, d]$ if and only if there exists a translation x of $\mathbb{R}^n$ such that $C + x \subseteq A$, $c + x = a$ and $d + x = b$, for all non-zero elements $[a, A, b]$ and $[c, C, d]$ in $S(\mathcal{T})$. Thus, in an inverse semigroup associated with a factorial language as in a one-dimensional tiling semigroup, we have $(i, w, j) \leqslant (k, u, l)$ if and only if u is a factor of w when we superimpose the i^{th} letter of w with the k^{th} letter of u and the j^{th} letter of w with the l^{th} letter of u, for all non-zero elements (i, w, j) and (k, u, l). In particular, for all non-zero idempotents (i, w, i) and (j, u, j) in a one-dimensional tiling semigroup, or in an inverse semigroup associated with a factorial language, we have $(i, w, i) \leqslant (j, u, j)$ if and only if u is a factor of w when we superimpose the i^{th} letter of w with the j^{th} letter of u. Therefore, $(i, w, i) \leqslant (j, u, j)$ if and only if there exist $x, y \in \Sigma^*$ such that $w = xuy$ and $i = |x| + j$.



Example 5.1.1. Suppose that *aabcac* is a word in a factorial language L. Then $\grave{a}ab\acute{c}ac$, $aa\acute{b}\grave{c}ac$ and $aabc\breve{a}c$ are non-zero elements in the semigroup $S(L)$. In the alternative representation we just introduced, we will write $(1, aabcac, 4)$ instead of $\grave{a}ab\acute{c}ac$, $(4, aabcac, 3)$ instead of $aa\acute{b}\grave{c}ac$, and $(5, aabcac, 5)$ instead of $aabc\breve{a}c$.

To compute the product $(1, aabcac, 4)\,(2, bcacc, 1)$ (assuming that $bcacc \in L$), we superimpose the fourth letter of *aabcac* with the second letter of *bcacc*, form the resulting word, which is *aabcacc*, and, assuming that *aabcacc* belongs to L, locate the position of the in-letter of the first element and the position of the out-letter of the second element in *aabcacc*. In this case, the first letter of *aabcac* is the first letter of *aabcacc* and the first letter of *bcacc* is the third letter of *aabcacc*. Thus, $(1, aabcac, 4)\,(2, bcacc, 1) = (1, aabcacc, 3)$. Of course, if *aabcacc* did not belong to L, then $(1, aabcac, 4)\,(2, bcacc, 1) = 0$.

Also, we have $(1, aabcacc, 4) \leqslant (1, aabcac, 4)$ but $(1, aabcacc, 3) \nleqslant (1, aabcac, 4)$ and $(1, aabcacc, 3) \nleqslant (2, abcac, 4)$. In the second case, it is not possible to simultaneously superimpose the first letter of *aabcacc* with the first letter of *aabcac* and the third letter of *aabcacc* with the fourth letter of *aabcac*; in the third case, although the relative position between the in-letter and the out-letter of the elements coincide, we have that *abcac* is not a factor of *aabcacc* when we superimpose the second letter of *abcac* with the first letter of *aabcac* (or the fourth letter of *abcac* with the third letter of *aabcac*).

As described in Section 1.1, all that is needed to build a representation of a strongly E^*-unitary inverse semigroup as a P^*-semigroup is a 0-restricted idempotent-pure pre-homomorphism from the semigroup into a group with zero.

Let $S(L)$ be the inverse semigroup associated with a factorial language L. The proof of Theorem 4.2.1 provides us with the following 0-restricted idempotent-pure pre-homomorphism $\lambda\colon S(L) \to \mathbb{Z}^{\overline{0}}$ from $S(L)$ into the abelian group with zero $\mathbb{Z}^{\overline{0}}$: for all $s \in S(L)$,

$$s\lambda = \begin{cases} j - i, & \text{if } s = (i, w, j) \\ \overline{0}, & \text{if } s = 0\,. \end{cases}$$

Following the construction described in Section 1.1, we begin by defining a preorder on $E_{S(L)} \times \mathbb{Z}$ by

$$(e, m) \preceq (f, n) \Leftrightarrow \exists\, s \in S(L) \text{ such that } e = ss^{-1}, s^{-1}s \leqslant f, \text{ and } -m + n \in \{s\lambda\}\,. \tag{5.1}$$

Let us simplify the expression of $\preceq$. For the non-zero idempotents of $S(L)$, we have

$$\begin{aligned} ((i, w, i), m) \preceq ((j, v, j), n) \Leftrightarrow \exists\, (k, u, l) \in S(L) & \text{ such that } (i, w, i) = (k, u, k), \\ & (l, u, l) \leqslant (j, v, j), \text{ and } -m + n = l - k\,. \end{aligned}$$

From $(k, u, k) = (i, w, i)$, we get that $k = i$ and $u = w$. From $(l, u, l) \leqslant (j, v, j)$, we get that there exist $x, y \in \Sigma^*$ such that $u = xvy$ and $|x| + j = l$, by the description of the natural partial order on $S(L)$. So, finally, $n = m + l - k = m + |x| + j - i$. Conversely, if there exist $x, y \in \Sigma^*$ such that $w = xvy$ and $n = m + l - k = m + |x| + j - i$, then we have $((i, w, i), m) \preceq ((j, v, j), n)$.

With respect to the idempotent 0, we have $(0, m) \preceq (f, n)$, for all $f \in E_{S(L)}$ and all $m, n \in \mathbb{Z}$, and $(e, m) \preceq (0, n)$ if and only if $e = 0$, for all $m, n \in \mathbb{Z}$.

Therefore, the rule

$$\begin{cases} (0, m) \preceq (f, n), \\ ((i, w, i), m) \preceq ((j, v, j), n) \Leftrightarrow \exists\, x, y \in \Sigma^* \text{ such that } w = xvy \text{ and } n = m + |x| + j - i\,, \end{cases}$$

for all $m, n \in \mathbb{Z}$, $f \in E_{S(L)}$ and $(i, w, i), (j, v, j) \in E_{S(L)} \setminus \{0\}$, is a quasi-order on $E_{S(L)} \times \mathbb{Z}$. Thus, we obtain an equivalence $\equiv$ on $E_{S(L)} \times \mathbb{Z}$ defined by, for all $m, n \in \mathbb{Z}$,

$$(0, m) \equiv (0, n)$$

and, for all $((i,w,i),m),((j,v,j),n) \in (E_{S(L)} \setminus \{0\}) \times \mathbb{Z}$,

$$\begin{aligned}((i,w,i),m) \equiv ((j,v,j),n) &\Leftrightarrow \begin{cases} \exists\, x,y \in \Sigma^* \text{ such that } w = xvy \text{ and } n = m + |x| + j - i \\ \exists\, x',y' \in \Sigma^* \text{ such that } v = x'wy' \text{ and } m = n + |x'| + i - j \end{cases} \\ &\Leftrightarrow w = v \text{ and } m - i = n - j\,.\end{aligned}$$

Let $\mathcal{U} = (E_{S(L)} \times \mathbb{Z})/\equiv$. Notice that, for $((i,w,i),m),((1,v,1),n) \in (E_{S(L)} \setminus \{0\}) \times \mathbb{Z}$,

$$\begin{aligned}((i,w,i),m) \equiv ((1,v,1),n) &\Leftrightarrow w = v \text{ and } m - i = n - 1 \\ &\Leftrightarrow w = v \text{ and } n = m - i + 1\,,\end{aligned}$$

and so each $\equiv$-class other than the minimum element of $\mathcal{U}$, which is $[0,0] = \{(0,m)\colon m \in \mathbb{Z}\}$, contains exactly one element of the form $((1,v,1),n)$. Therefore, we may identify the elements of $\mathcal{U} \setminus \{[0,0]\}$ with the elements of $\mathbb{Z} \times L$. Moreover, *under this identification*, we have for all $(m,w),(n,v) \in \mathbb{Z} \times L$,

$$\begin{aligned}(m,w) \leqslant (n,v) &\Leftrightarrow [(1,w,1),m] \leqslant [(1,v,1),n] \\ &\Leftrightarrow ((1,w,1),m) \preceq ((1,v,1),n) \\ &\Leftrightarrow \exists\, x,y \in \Sigma^* \text{ such that } w = xvy \text{ and } n = m + |x| + 1 - 1 \\ &\Leftrightarrow \exists\, x,y \in \Sigma^* \text{ such that } w = xvy \text{ and } |x| = n - m\,. \end{aligned} \tag{5.2}$$

Now let $\mathcal{V} = \big\{[e,0]\colon e \in E_{S(L)}\big\}$. Then,

$$\begin{aligned}\mathcal{V} &= \big\{[(i,w,i),0]\colon (i,w,i) \in E_{S(L)} \setminus \{0\}\big\} \cup \{[0,0]\} \\ &= \{[(1,w,1),0-i+1]\colon w \in L \text{ and } 1 \leqslant i \leqslant |w|\} \cup \{[0,0]\} \\ &\simeq \{(1-i,w)\colon w \in L \text{ and } 1 \leqslant i \leqslant |w|\} \cup \{0\} \\ &= \{(j,w) \in \mathbb{Z} \times L\colon 1 - |w| \leqslant j \leqslant 0\} \cup \{0\}\ . \end{aligned} \tag{5.3}$$

Also, the law

$$\begin{cases} k \cdot 0 = 0 \\ k \cdot (m,w) = (k+m,w)\,, \end{cases} \tag{5.4}$$

for $k \in \mathbb{Z}$ and $(m,w) \in \mathbb{Z} \times L$, defines the action of the group $\mathbb{Z}$ on $\mathcal{U} = (\mathbb{Z} \times L) \cup \{0\}$.

These ingredients give us the P-semigroup

$$\begin{aligned}P(\mathbb{Z},\mathcal{U},\mathcal{V}) &= \{(x,k) \in \mathcal{V} \times \mathbb{Z}\colon (-k)\cdot x \in \mathcal{V}\} \\ &= \{((m,w),k) \in (\mathcal{V} \setminus \{0\}) \times \mathbb{Z}\colon (-k)\cdot(m,w) \in \mathcal{V}\} \cup \{(0,k)\colon (-k)\cdot 0 \in \mathcal{V}\} \\ &= \{((m,w),k) \in (\mathcal{V} \setminus \{0\}) \times \mathbb{Z}\colon (m-k,w) \in \mathcal{V}\} \cup \{(0,k)\colon k \in \mathbb{Z}\} \\ &= \{((m,w),k) \in (\mathbb{Z} \times L) \times \mathbb{Z}\colon 1 - |w| \leqslant m \leqslant 0 \text{ and } 1 - |w| \leqslant m - k \leqslant 0\} \\ &\qquad \cup \{(0,k)\colon k \in \mathbb{Z}\}\,. \end{aligned} \tag{5.5}$$

By Construction 1.1.4, we know that $S(L) \simeq P^*(\mathbb{Z},\mathcal{U}^*,\mathcal{V}^*)$, with $\mathcal{U}^* = \mathcal{U} \setminus \{0\}$ and $\mathcal{V}^* = \mathcal{V} \setminus \{0\}$, and where $\mathcal{U}$ and $\mathcal{V}$ and the action of $\mathbb{Z}$ on $\mathcal{U}$ are as above. That is,

$$S(L) \simeq \{(m,w,k) \in \mathbb{Z} \times L \times \mathbb{Z} \colon 1-|w| \leqslant m \leqslant 0 \text{ and } 1-|w| \leqslant m-k \leqslant 0\} \cup \{0\}\,.$$

Notice that, whereas the non-zero idempotents of $S(L)$ are characterized by a word w in L and a non-negative integer i between 1 and $|w|$, the non-zero elements of $\mathcal{V}$ are pairs (j,w) with $w \in L$ and j an integer between $1-|w|$ and 0. In order to obtain a more natural definition for the elements of the semilattice $\mathcal{V}$ — recall that $\mathcal{V}$ and $E_{S(L)}$ are isomorphic — we will consider a slightly different P-semigroup $P(\mathbb{Z},\mathcal{X},\mathcal{Y})$, designed to be isomorphic to $P(\mathbb{Z},\mathcal{U},\mathcal{V})$. Consequently, the corresponding semigroup $P(\mathbb{Z},\mathcal{X}^*,\mathcal{Y}^*)$ will yield a simpler representation of $S(L)$ as a P^*-semigroup.

As known, isomorphisms of P-semigroups are obtained from compatible isomorphisms between the groups and order isomorphisms between the partially ordered sets. Consider the set $\mathcal{X} = (\mathbb{Z} \times L) \cup \{0\}$. The map $\Phi \colon \mathcal{U} \to \mathcal{X}$ defined by $0\Phi = 0$ and $(m,w)\Phi = (1-m,w)$ is easily seen to be a bijection, with inverse $\Phi^{-1} \colon \mathcal{X} \to \mathcal{U}$ defined by $0\Phi = 0$ and $(m,w)\Phi^{-1} = (1-m,w)$. Let $\mathcal{Y} = \mathcal{V}\Phi = \{(q,w)\Phi \in \mathbb{Z} \times L \colon 1-|w| \leqslant q \leqslant 0\} \cup \{0\}$, that is

$$\begin{aligned}\mathcal{Y} &= \{(1-q,w) \in \mathbb{Z} \times L \colon 1-|w| \leqslant q \leqslant 0\} \cup \{0\} \\ &= \{(m,w) \in \mathbb{Z} \times L \colon 1 \leqslant m \leqslant |w|\} \cup \{0\}\ . \end{aligned} \tag{5.6}$$

Now equip $\mathcal{X}$ with the order defined by $0 \leqslant (m,w)$ and

$$\begin{aligned}(m,w) \leqslant (n,v) &\Leftrightarrow (m,w)\Phi^{-1} \leqslant (n,v)\Phi^{-1} \\ &\Leftrightarrow (1-m,w) \leqslant (1-n,v) \\ &\Leftrightarrow \exists\, x,y \in \Sigma^* \text{ such that } w = xvy \text{ and } |x| = (1-n)-(1-m) \\ &\Leftrightarrow \exists\, x,y \in \Sigma^* \text{ such that } w = xvy \text{ and } |x| = m-n\,, \end{aligned} \tag{5.7}$$

for all $(m,w),(n,v) \in \mathcal{X}$. Consequently, Φ is an order isomorphism between the partially ordered sets $\mathcal{U}$ and $\mathcal{X}$. Similarly, we can use Φ and Φ^{-1} to define an action of $\mathbb{Z}$ on $\mathcal{X}$ with respect to which $(id_{\mathbb{Z}},\Phi)$ is a compatible pair. As we wish to require that

$$k \cdot 0 = (k \cdot (0\Phi^{-1}))\Phi = (k \cdot 0)\Phi = 0\Phi = 0$$

and

$$\begin{aligned}k \cdot (m,w) &= (k \cdot ((m,w)\Phi^{-1})\Phi \\ &= (k \cdot (1-m,w))\Phi \\ &= (k+1-m,w)\Phi \\ &= (1-(k+1-m),w) \\ &= (m-k,w)\,,\end{aligned}$$

we just define

$$\begin{cases} k \cdot 0 = 0 \\ k \cdot (m, w) = (m - k, w) \quad \text{for } (m, w) \in \mathbb{Z} \times L\,. \end{cases} \tag{5.8}$$

Thus, by Theorem 1.1.2 and the fact that $S(L) \simeq P^*(\mathbb{Z}, \mathcal{U}^*, \mathcal{V}^*)$, we have:

Theorem 5.1.2. *Let L be a factorial language over a finite alphabet Σ. Let $\mathcal{X} = (\mathbb{Z} \times L) \cup \{0\}$ and*

$$\mathcal{Y} = \{(m, w) \in \mathcal{X} \colon 1 \leqslant m \leqslant |w|\} \cup \{0\}\ .$$

Then, defining $0 \leqslant (m, w)$ and

$$(m, w) \leqslant (n, v) \Leftrightarrow \exists\, x, y \in \Sigma^* \text{ such that } w = xvy \text{ and } |x| = m - n,$$

for all (m, w), $(n, v) \in \mathcal{X}$, and the action of $\mathbb{Z}$ on $\mathcal{X}$ by $k \cdot 0 = 0$ and $k \cdot (m, w) = (m - k, w)$, for all $k \in \mathbb{Z}$ and $(m, w) \in \mathcal{X}$, then $(\mathbb{Z}, \mathcal{X}, \mathcal{Y})$ is a McAlister 0-triple and $S(L) \simeq P^*(\mathbb{Z}, \mathcal{X}^*, \mathcal{Y}^*)$, where

$$P^*(\mathbb{Z}, \mathcal{X}^*, \mathcal{Y}^*) = \{((m, w), k) \in (\mathbb{Z} \times L) \times \mathbb{Z} \colon 1 \leqslant m \leqslant |w| \text{ and } 1 \leqslant m + k \leqslant |w|\} \cup \{0\}$$

and

$$((i, w), j)((m, v), k) = \begin{cases} ((i, w) \wedge (m - j, v), j + k), & \text{if } (i, w) \wedge (m - j, v) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

5.2 Representation of an n-dimensional hypercubic tiling semigroup as a P*-semigroup

We now generalize the description given in the previous section to an inverse semigroup associated with an n-dimensional factorial language.

Recall from Section 4.1 that, given a positive integer n and a finite alphabet Σ, we say that L is an n-dimensional factorial language over Σ if it is a set of finite, non-empty and connected Σ-coloured subsets (A, α) of $\mathbb{Z}^n$ with $A_0 = 0$ such that for each (A, α) in L and each connected $B \subseteq A$, we have that $(B, \alpha|_B) - B_0$ belongs to L as well.

Also recall, from Section 3.4, that, given two Σ-coloured subsets (A, α) and (B, β) of $\mathbb{Z}^n$, we have $(A, \alpha) \leqslant (B, \beta)$ if and only if $A \subseteq B$ and $\beta|_A = \alpha$, and that, consequently, we often omit any reference to the mappings.

The following is the main result of this chapter.

Theorem 5.2.1. *Let L be an n-dimensional factorial language. Let $\mathcal{X} = (\mathbb{Z}^n \times L) \cup \{0\}$ and $\mathcal{Y} = \{(p, P) \in \mathcal{X} \setminus \{0\} \colon p \in P\} \cup \{0\}$. Then, defining $0 \leqslant (p, P)$ and*

$$(p, P) \leqslant (q, Q) \Leftrightarrow Q - q \leqslant P - p\,,$$

for all $(p,P),(q,Q) \in \mathcal{X} \setminus \{0\}$*, and the action of* $\mathbb{Z}^n$ *on* $\mathcal{X}$ *by* $u \cdot 0 = 0$ *and* $u \cdot (p,P) = (p-u,P)$*, for all* $u \in \mathbb{Z}^n$ *and* $(p,P) \in \mathcal{X} \setminus \{0\}$*, we have that* $(\mathbb{Z}^n,\mathcal{X},\mathcal{Y})$ *is a McAlister* 0*-triple and* $S(L) \simeq P^*(\mathbb{Z}^n,\mathcal{X}^*,\mathcal{Y}^*)$*, with*

$$P^*(\mathbb{Z}^n,\mathcal{X}^*,\mathcal{Y}^*) = \{((p,P),u) \in (\mathbb{Z}^n \times L) \times \mathbb{Z}^n \colon p, p+u \in P\} \cup \{0\}$$

and

$$((p,P),u)((q,Q),v) = \begin{cases} ((p,P) \wedge (q-u,Q), u+v), & \textit{if } (p,P) \wedge (q-u,Q) \neq 0 \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Although much of the proof is direct checking, here we include the details, as only a sketch of it appears in [44].

Claim 1. $(\mathbb{Z}^n,\mathcal{X},\mathcal{Y})$ is a McAlister 0-triple.

We first check that $(\mathcal{X},\leqslant)$ is a partially ordered set (observe that 0 is, by definition, the minimum element of $\mathcal{X}$). The binary relation $\leqslant$ defined on $\mathcal{X}$ is trivially reflexive and transitive. Let $(p,P),(q,Q) \in \mathcal{X}$ be such that $(p,P) \leqslant (q,Q)$ and $(q,Q) \leqslant (p,P)$. Then $Q-q \leqslant P-p$ and $P-p \leqslant Q-q$ (as Σ-coloured subsets), and so $P-p = Q-q$. Since $P_0 = Q_0 = 0$ by definition of n-dimensional factorial language, then

$$-p = P_0 - p = (P-p)_0 = (Q-q)_0 = Q_0 - q = -q\,,$$

that is, $p = q$, and so $P = Q - q + p = Q$. Therefore, $(P,p) = (Q,q)$ and so $\leqslant$ is anti-symmetric. Thus, $(\mathcal{X},\leqslant)$ is a down directed partially ordered set.

To check that $\mathcal{Y}$ is an order ideal of $\mathcal{X}$, let $(p,P) \in \mathcal{Y}$ and $(q,Q) \in \mathcal{X}$ be such that $(q,Q) \leqslant (p,P)$. Since $p \in P$ by definition of $\mathcal{Y}$ and $P-p \subseteq Q-q$ by definition of $\leqslant$, then $0 = p-p \in P-p \subseteq Q-q$, whence $q \in Q$ and so $(Q,q) \in \mathcal{Y}$. Therefore, $\mathcal{Y}$ is an order ideal of $\mathcal{X}$.

We now show that $\mathcal{Y}$ is an inf-subsemilattice of $\mathcal{X}$. Let $(p,P),(q,Q) \in \mathcal{Y}$.

We claim that in case there does not exist $(r,R) \in \mathcal{X} \setminus \{0\}$ such that $(r,R) \leqslant (p,P)$ or $(r,R) \leqslant (q,Q)$, then $(p,P) \wedge (q,Q) = 0$. In fact, since by definition 0 is the minimum element of $\mathcal{X}$, then $0 \leqslant (p,P)$ and $0 \leqslant (q,Q)$. Therefore, if there does not exist $(r,R) \in \mathcal{X}$ such that $(r,R) \leqslant (p,P)$ and $(r,R) \leqslant (q,Q)$, then $(p,P) \wedge (q,Q) = 0$ by definition of infimum.

Next assume there exists $(r,R) \in \mathcal{X} \setminus \{0\}$ such that $(r,R) \leqslant (p,P)$ and $(r,R) \leqslant (q,Q)$. We claim that $(p,P) \wedge (q,Q) = (p \vee q, ((P-p) \cup (Q-q)) + p \vee q) \in \mathcal{Y}$. We begin by showing that $((P-p) \cup (Q-q)) + p \vee q \in L$. By definition of $\leqslant$, we have $P-p \subseteq R-r$ and $Q-q \subseteq R-r$. Thus, $(P-p) \cup (Q-q) \subseteq R-r$, or, equivalently, $((P-p) \cup (Q-q)) + r \subseteq R$. As $0 \in (P-p) \cap (Q-q)$ and P and Q are connected, we have that $(P-p) \cup (Q-q)$ and, consequently, $((P-p) \cup (Q-q)) + r$ are connected. Therefore, since $R \in L$ by definition of $\mathcal{X}$ and as L is factorial by assumption, we may conclude that $((P-p)\cup(Q-q))+r-(((P-p)\cup(Q-q))+r)_0\in L$. Now,

$$\begin{aligned}
&((P-p)\cup(Q-q))+r-((P-p)\cup(Q-q))_0-r\in L\Rightarrow\\
&\quad\Rightarrow((P-p)\cup(Q-q))-\min((P-p)_0,(Q-q)_0)\in L\\
&\quad\Rightarrow((P-p)\cup(Q-q))-\min(P_0-p,Q_0-q)\in L\\
&\quad\Rightarrow((P-p)\cup(Q-q))-\min(-p,-q)\in L\\
&\quad\Rightarrow((P-p)\cup(Q-q))+\max(p,q)\in L\\
&\quad\Rightarrow((P-p)\cup(Q-q))+p\vee q\in L\,,
\end{aligned}$$

as desired. Further, since $p\vee q\in((P-p)\cup(Q-q))+p\vee q$ as $0=p-p\in P-p$, then $(p\vee q,((P-p)\cup(Q-q))+p\vee q)\in\mathcal{X}$. Moreover, $P-p\subseteq(P-p)\cup(Q-q)$ and $Q-q\subseteq(P-p)\cup(Q-q)$ imply that

$$\begin{cases}P-p\subseteq((P-p)\cup(Q-q))+p\vee q-p\vee q\\ Q-q\subseteq((P-p)\cup(Q-q))+p\vee q-p\vee q\,.\end{cases}$$

It follows that

$$\begin{cases}(p\vee q,((P-p)\cup(Q-q))+p\vee q)\leqslant(p,P)\\ (p\vee q,((P-p)\cup(Q-q))+p\vee q)\leqslant(q,Q)\,.\end{cases}$$

Also, if $(t,T)\in\mathcal{X}$ is such that $(t,T)\leqslant(p,P)$ and $(t,T)\leqslant(q,Q)$, then $P-p\subseteq T-t$ and $Q-q\subseteq T-t$. Thus, $(P-p)\cup(Q-q)\subseteq T-t$, or, equivalently,

$$((P-p)\cup(Q-q))+p\vee q-p\vee q\subseteq T-t\,.$$

So, $(t,T)\leqslant(p\vee q,((P-p)\cup(Q-q))+p\vee q)$. Therefore,

$$(p,P)\wedge(q,Q)=(p\vee q,((P-p)\cup(Q-q))+p\vee q)\,,$$

by definition of infimum.

Since $\mathcal{Y}$ is an order ideal of $\mathcal{X}$, the fact that $(p\vee q,((P-p)\cup(Q-q))+p\vee q)\leqslant(p,P)$ implies that $(p\vee q,((P-p)\cup(Q-q))+p\vee q)\in\mathcal{Y}$. Hence, $\mathcal{Y}$ is an inf-subsemilattice of $\mathcal{X}$, with $0\wedge 0=0$, $(p,P)\wedge 0=0$, $0\wedge(p,P)=0$, and

$$(p,P)\wedge(q,Q)=\begin{cases}(p\vee q,((P-p)\cup(Q-q))+p\vee q) & \text{if } ((P-p)\cup(Q-q))+p\vee q\in L\\ 0 & \text{otherwise}\,,\end{cases}$$

for all $(p,P),(q,Q)\in\mathcal{Y}$.

Let us now show that the rule

$$\begin{cases}u\cdot 0=0\\ u\cdot(p,P)=(p-u,P)\quad\text{with } (p,P)\in\mathcal{X}\setminus\{0\}\end{cases}$$

for each $u \in \mathbb{Z}^n$, defines an action of $\mathbb{Z}^n$ on the left on $\mathcal{X}$ by order automorphisms. Let $u \in \mathbb{Z}^n$ and consider the map $u\colon \mathcal{X} \to \mathcal{X}$ defined by $u \cdot 0 = 0$ and $u \cdot (p, P) = (p - u, P)$, for all $(p, P) \in \mathcal{X}$. Since $p - u \in \mathbb{Z}^n$, then $(p - u, P) \in \mathcal{X}$, and so the mapping is well-defined. Now, $0 \leqslant (q, Q)$ and $u \cdot 0 = 0 \leqslant u \cdot (q, Q)$, trivially. Also, if $(p, P) \leqslant (q, Q)$ in $\mathcal{X}$, then $Q - q \leqslant P - p$ and so $Q - q + u \leqslant P - p + u$, that is, $Q - (q - u) \leqslant P - (p - u)$. So, $u \cdot (p, P) \leqslant u \cdot (q, Q)$. Thus, we may conclude that the map defined by u is an order homomorphism, since it respects the order of $\mathcal{X}$. Futher, $(u + v) \cdot 0 = 0 = u \cdot 0 = u \cdot (v \cdot 0)$ and

$$(u+v)\cdot(p,P) = (p-(u+v),P) = (p-u-v,P) = ((p-v)-u,P) = u\cdot(p-v,P) = u\cdot(v\cdot(p,P))\,,$$

for all $u, v \in \mathbb{Z}^n$ and $(p, P) \in \mathcal{X}$. As $0 \cdot (p, P) = (p - 0, P) = (p, P)$, for all $(p, P) \in \mathcal{X}$, we may conclude that u is one-one. Therefore, $\mathbb{Z}^n$ acts on the left on $\mathcal{X}$ by order automorphisms.

Finally, we check that $\mathbb{Z}^n \cdot \mathcal{Y} = \mathcal{X}$. Evidently, $\mathbb{Z}^n \cdot \mathcal{Y} \subseteq \mathcal{X}$, as $\mathcal{Y} \subseteq \mathcal{X}$ and $\mathbb{Z}^n \cdot \mathcal{X} \subseteq \mathcal{X}$. Conversely, let $(q, Q) \in \mathcal{X}$. Then $Q \in L$ by definition of $\mathcal{X}$ and so $0 \in Q$ by definition of factorial language. Thus, $(0, Q) \in \mathcal{Y}$ is such that $(-q) \cdot (0, Q) = (0 - (-q), Q) = (q, Q)$, with $-q \in \mathbb{Z}^n$. Therefore, $(q, Q) \in \mathbb{Z}^n \cdot \mathcal{Y}$, as required. We conclude that $\mathcal{X} \subseteq \mathbb{Z}^n \cdot \mathcal{Y}$. Hence, $\mathbb{Z}^n \cdot \mathcal{Y} = \mathcal{X}$.

It follows that $(\mathbb{Z}^n, \mathcal{X}, \mathcal{Y})$ is a McAlister 0-triple.

By definition of $P^*(\mathbb{Z}^n, \mathcal{X}^*, \mathcal{Y}^*)$, we have

$$\begin{aligned}
P^*(\mathbb{Z}^n, \mathcal{X}^*, \mathcal{Y}^*) &= \{((p, P), u) \in \mathcal{Y}^* \times \mathbb{Z}^n \colon (-u) \cdot (p, P) \in \mathcal{Y}^*\} \cup \{0\} \\
&= \{((p, P), u) \in \mathcal{X}^* \times \mathbb{Z}^n \colon (p, P), (p + u, P) \in \mathcal{Y}^*\} \cup \{0\} \\
&= \{((p, P), u) \in (\mathbb{Z}^n \times L) \times \mathbb{Z}^n \colon p, p + u \in P\} \cup \{0\}\ ,
\end{aligned}$$

where $\mathcal{X}^* = \mathcal{X} \setminus \{0\}$, $\mathcal{Y}^* = \mathcal{Y} \setminus \{0\}$.

Claim 2. $S(\mathcal{T}) \simeq P^*(\mathbb{Z}^n, \mathcal{X}^*, \mathcal{Y}^*)$.

Let $\pi\colon S(L) \to P^*(\mathbb{Z}^n, \mathcal{X}^*, \mathcal{Y}^*)$ be the mapping defined by $0\pi = 0$ and, for each (p, P, q) in $S(L) \setminus \{0\}$, by $(p, P, q)\pi = ((p, P), q - p)$. It is trivial to check that π is well-defined and injective. To see that it is onto, simply note that if $((p, P), u) \in P^*(\mathbb{Z}^n, \mathcal{X}^*, \mathcal{Y}^*)$, then p and $p + u$ belong to P, so that $(p, P, p + u) \in S(L)$ is such that

$$(p, P, p + u)\pi = ((p, P), p + u - p) = ((p, P), u)\,.$$

Thus, π is bijective.

To complete the proof, we show that π is a homomorphism. Clearly, $(0 * 0)\pi = 0 = 0\pi\, 0\pi$ and $((p, P, q) * 0)\pi = 0 = (p, P, q)\pi\, 0\pi$ and $(0 * (p, P, q))\pi = 0 = 0\pi\, (p, P, q)\pi$, for all (p, P, q)

in $S(L)$. Let $(p,P,q),(r,R,s)\in S(L)$. On the one hand, we have

$$((p,P,q)*(r,R,s))\pi =$$

$$= \begin{cases} (p-q+q\vee r, ((P-q)\cup(R-r))+q\vee r, s-r+q\vee r)\pi \\ \qquad\qquad\qquad\qquad\qquad \text{if } (P-q)\cup(R-r))+q\vee r\in L \\ 0\pi \qquad\qquad\qquad\qquad \text{otherwise} \end{cases}$$

$$= \begin{cases} ((p-q+q\vee r, ((P-q)\cup(R-r))+q\vee r), s-r+q\vee r-(p-q+q\vee r)) \\ \qquad\qquad\qquad\qquad\qquad \text{if } (P-q)\cup(R-r))+q\vee r\in L \\ 0 \qquad\qquad\qquad\qquad \text{otherwise} \end{cases}$$

$$= \begin{cases} ((p-q+q\vee r, ((P-q)\cup(R-r))+q\vee r), s-r+q-p) \\ \qquad\qquad\qquad\qquad\qquad \text{if } (P-q)\cup(R-r))+q\vee r\in L \\ 0 \qquad\qquad\qquad\qquad \text{otherwise ;} \end{cases}$$

on the other hand,

$$(p,P,q)\pi\,(r,R,s)\pi = ((p,P),q-p)((r,R),s-r) =$$

$$= \begin{cases} ((p,P)\wedge(r-(q-p),R), q-p+s-r) & \text{if } (p,P)\wedge(r-(q-p),R)\neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} ((p\vee(r-q+p), ((P-p)\cup(R-(r-q+p)))+p\vee(r-q+p)), s-r+q-p) \\ \qquad\qquad\qquad \text{if } ((P-p)\cup(R-(r-q+p))+p\vee(r-q+p)\in L \\ 0 \qquad\qquad\qquad \text{otherwise .} \end{cases}$$

Since $p\vee(r-q+p)=(p-q+q)\vee(r+p-q)=p-q+q\vee r$, then

$$(p,P,q)\pi\,(r,R,s)\pi =$$

$$= \begin{cases} ((p-q+q\vee r, ((P-p)\cup(R-r+q-p))+p-q+q\vee r), s-r+q-p) \\ \qquad\qquad\qquad \text{if } ((P-p)\cup(R-r+q-p)+p-q+q\vee r\in L \\ 0 \qquad\qquad\qquad \text{otherwise} \end{cases}$$

$$= \begin{cases} ((p-q+q\vee r, ((P-p+p-q)\cup(R-r+q-p+p-q)+q\vee r), s-r+q-p) \\ \qquad\qquad\qquad \text{if } ((P-p+p-q)\cup(R-r+q-p+p-q))+q\vee r\in L \\ 0 \qquad\qquad\qquad \text{otherwise} \end{cases}$$

$$= \begin{cases} ((p-q+q\vee r, ((P-q)\cup(R-r))+q\vee r), s-r+q-p) \\ \qquad\qquad\qquad \text{if } ((P-q)\cup(R-r))+q\vee r\in L \\ 0 \qquad\qquad\qquad \text{otherwise .} \end{cases}$$

Therefore, $(p, P, q)\pi\,(r, R, s)\pi = ((p, P, q) * (r, R, s))\pi$. It follows that π is a homomorphism.

Hence, $S(L) \simeq P^*(\mathbb{Z}^n, \mathcal{X}^*, \mathcal{Y}^*)$. □

Remark 5.2.2. Notice how the P^*-representation of the inverse semigroup associated with an arbitrary n-dimensional factorial language described in the previous theorem exactly generalizes the representation constructed for the one-dimensional case in the previous section. In fact, let L be a one-dimensional factorial language.

1. It is easy to see that Theorem 5.2.1 is the same statement as Theorem 5.1.2 when $n = 1$. First of all, the traditional notion of factorial language is equivalent to the notion of one-dimensional factorial language given by Definition 4.1.3 and the inverse semigroup considered by Lawson coincides with the inverse semigroup associated with a one-dimensional factorial language from Theorem 4.2.1, as noticed in Remark 4.2.4. Evidently, $\mathcal{X}$ and the action of $\mathbb{Z}$ on $\mathcal{X}$ are the same. The definition of $\mathcal{Y}$ and of $P^*(\mathbb{Z}, \mathcal{X}^*, \mathcal{Y}^*)$ in both results are obviously equivalent, since singling out a letter of a word is the same as indicating its position. By the comments about the natural partial order on the idempotents of $S(L)$ made at the beginning of Section 5.1, we conclude that both results, Theorem 5.1.2 and Theorem 5.2.1 for $n = 1$, consider the same partial order on $\mathcal{X}$, since the condition "$Q - q \leqslant P - p$" requires that $Q - q$ is a subset of $P - p$ and that the colours match when p and q are shifted to 0 (or, equivalently, when p and q are superimposed). Since the action and the order coincide, so does the operation defined on the P^*-semigroups.

2. Consequently, we can read off from the proof of Theorem 5.2.1 what the isomorphism $\pi\colon S(L) \to P^*(\mathbb{Z}, \mathcal{X}^*, \mathcal{Y}^*)$ is: $0\pi = 0$ and, for all non-zero element $(i, w, j) \in S(L)$,

$$(i, w, j)\pi = ((i, w), j - i)\,.$$

5.3 Some examples

In this final section we examine some examples of isomorphic P^*-semigroups that deal with the P^*-representation previously obtained for inverse semigroups associated with one-dimensional factorial languages.

In our first example, we exhibit two isomorphic P^*-semigroups where the groups involved are not isomorphic.

Example 5.3.1. Let L be a factorial language over an alphabet Σ. Recall from Section 3.4 that $S(L)$ is generated, as an inverse semigroup with zero, by its one-letter idempotents (that is, the elements of the form $\check{a}$, with $a \in \Sigma$ or, equivalently, $a \in L$), and its two-letter elements of the form $\grave{a}_i\acute{a}_j$, with $a_i a_j \in L$.

Let X be the set of two-letter words in L, that is, $X = \{ab \in L \colon a, b \in \Sigma\}$, and let FG_X be the free group on X. Consider the map $\lambda\colon S(L) \to {FG_X}^{\overline{0}}$ defined on the non-zero elements by $0\lambda = \overline{0}$ and

$$(i, a_1 \dots a_k, j)\lambda = \begin{cases} (a_i a_{i+1})(a_{i+1} a_{i+2}) \dots (a_{j-1} a_j) & \text{if } i < j \\ 1 & \text{if } i = j \\ [(a_j a_{j+1})(a_{j+1} a_{j+2}) \dots (a_{i-1} a_i)]^{-1} & \text{if } i > j\,. \end{cases}$$

It is clear that λ is 0-restricted and idempotent-pure. Since $(1, a, 1)\lambda = 1$ and $(1, ab, 2)\lambda = ab$, for all $ab \in L$, it follows that FG_X is generated as a group by $S(L)\lambda$. As $(1, ab, 2)\,(1, cd, 2) \neq 0$ if and only if $b = c$ and $abd \in L$, in which case

$$((1, ab, 2)\,(1, cd, 2))\lambda = (1, abd, 3)\lambda = (ab)(bd) = (1, ab, 2)\lambda\,(1, cd, 2)\lambda\,,$$

we have that $(1, ab, 2)\,(1, cd, 2) \neq 0$ implies that $((1, ab, 2)\,(1, cd, 2))\lambda = (1, ab, 2)\lambda(1, cd, 2)\lambda$, for all $ab, cd \in L$. Since $S(L)$ is generated, as an inverse semigroup, by the one-letter idempotents $(1, a, 1)$ with $a \in L$ and the two-letter elements $(1, ab, 2)$ with $ab \in L$, we conclude that λ is a pre-homomorphism.

By Chapter 1 we obtain a P^*-semigroup $P^*(FG_X, \mathcal{Z}^*, \mathcal{W}^*)$ isomorphic to $S(L)$. On the other hand, $S(L) \simeq P^*(\mathbb{Z}, \mathcal{X}^*, \mathcal{Y}^*)$ by Theorem 5.1.2. However, FG_X and $\mathbb{Z}$ are not isomorphic, unless L has a unique two-letter word. In particular, the corresponding P-semigroups $P(FG_X, \mathcal{Z}, \mathcal{W})$ and $P(\mathbb{Z}, \mathcal{X}, \mathcal{Y})$ cannot be isomorphic for the same reason (cf. Theorem 1.1.2).

The next example uses the P^*-representation of an inverse semigroup associated with a factorial language, obtained in Theorem 5.1.2, to prove an isomorphism between two inverse semigroups associated with factorial languages. The isomorphism between the P^*-semigroups will come from an isomorphism between the corresponding P-semigroups (cf. Theorem 1.1.2).

Example 5.3.2. Let L be a factorial language. Then L^{op}, the language of the reversals $w^{op} = a_l \dots a_1$ of the words $w = a_1 \dots a_l \in L$, is a factorial language as well. We will prove that $S(L)$ and $S(L^{op})$ are isomorphic.

Consider the P^*-representations $P^*(\mathbb{Z}, \mathcal{X}^*, \mathcal{Y}^*)$ and $P^*(\mathbb{Z}, \mathcal{X}^*_{op}, \mathcal{Y}^*_{op})$ of $S(L)$ and $S(L^{op})$, respectively, as in Theorem 5.1.2. Consider also the corresponding P-semigroups $P(\mathbb{Z}, \mathcal{X}, \mathcal{Y})$ and $P(\mathbb{Z}, \mathcal{X}_{op}, \mathcal{Y}_{op})$ defined by (5.6), (5.7) and (5.8).

We begin by constructing an isomorphism from $P(\mathbb{Z}, \mathcal{X}, \mathcal{Y})$ to $P(\mathbb{Z}, \mathcal{X}_{op}, \mathcal{Y}_{op})$. Let $\Phi\colon \mathcal{X} \to \mathcal{X}_{op}$ be the map defined by $0\Phi = 0$ and $(m, w)\Phi = (|w| - m + 1, w^{op})$. Clearly, Φ is well-defined and is a bijection. Also, since $w = xvy$ if and only if $w^{op} = y^{op}v^{op}x^{op}$ and, in this case, $|x| = m - n$ if and only if

$$\begin{aligned} |y^{op}| = |w^{op}| - |v^{op}| - |x^{op}| = |w| - |v| - |x| = \\ = |w| - |v| - (m - n) = (|w| - m + 1) - (|v| - n + 1)\,, \end{aligned}$$

we have

$$\begin{aligned}(m,w) \leqslant (n,v) &\Leftrightarrow \exists\, x,y \in \Sigma^* \text{ such that } w = xvy \text{ and } |x| = m-n\\ &\Leftrightarrow \exists\, x,y \in \Sigma^* \text{ such that } w^{op} = y^{op}v^{op}x^{op}\\ &\qquad\qquad \text{and } |y^{op}| = (|w|-m+1) - (|v|-n+1)\\ &\Leftrightarrow (|w|-m+1, w^{op}) \leqslant (|v|-n+1, v^{op})\\ &\Leftrightarrow (m,w)\Phi \leqslant (n,v)\Phi\,,\end{aligned}$$

for all $(m,w),(n,v) \in \mathcal{X}$. This shows that Φ is an order isomorphism. Further,

$$\begin{aligned}\mathcal{Y}\Phi &= \{(m,w)\Phi \in \mathbb{Z}\times L \colon 1 \leqslant m \leqslant |w|\} \cup \{0\Phi\}\\ &= \{(|w|-m+1, w^{op}) \in \mathbb{Z}\times L^{op} \colon 1 \leqslant m \leqslant |w|\} \cup \{0\}\\ &= \{(n, w^{op}) \in \mathbb{Z}\times L^{op} \colon 1 \leqslant n \leqslant |w|\} \cup \{0\}\\ &= \mathcal{Y}_{op}\,,\end{aligned}$$

since $1 \leqslant m \leqslant |w|$ if and only if $1 \leqslant |w|-m+1 \leqslant |w|$ and $|w| = |w^{op}|$.

Finally, we need a group isomorphism $\alpha\colon \mathbb{Z} \to \mathbb{Z}$ such that $(k\cdot 0)\Phi = (k\alpha)\cdot 0\Phi$ and $(k\cdot(m,w))\Phi = (k\alpha)\cdot(m,w)\Phi$, for all $k \in \mathbb{Z}$ and $(m,w) \in \mathcal{X}$, in order to have an isomorphism of P-semigroups. Since $(k\alpha)\cdot 0\Phi = (k\alpha)\cdot 0 = 0$ and

$$\begin{cases}(k\cdot(m,w))\Phi = (m-k,w)\Phi = (|w|-(m-k)+1, w^{op}) = (|w|-m+1+k, w^{op})\\ (k\alpha)\cdot(m,w)\Phi = (k\alpha)\cdot(|w|-m+1, w^{op}) = (|w|-m+1-k\alpha, w^{op})\,,\end{cases}$$

we have the desired conclusion for $k\alpha = -k$. (It would be equally easy to see that this is the only of the two automorphisms of $\mathbb{Z}$ that works.) Thus, by Theorem 1.1.2, the map $\psi\colon P(\mathbb{Z},\mathcal{X},\mathcal{Y}) \to P(\mathbb{Z},\mathcal{X}_{op},\mathcal{Y}_{op})$ defined by $0\psi = 0$ and, for all $((m,w),k) \in P(\mathbb{Z},\mathcal{X},\mathcal{Y})$,

$$((m,w),k)\psi = ((m,w)\Phi, k\alpha) = ((|w|+1-m, w^{op}), -k)\,,$$

is an isomorphism.

By an abuse of notation, identify ψ with the corresponding isomorphism between the P^*-semigroups $P^*(\mathbb{Z},\mathcal{X}^*,\mathcal{Y}^*)$ and $P^*(\mathbb{Z},\mathcal{X}^*_{op},\mathcal{Y}^*_{op})$, that maps 0 to 0 and a non-zero element $((m,w),k)$ to $((|w|+1-m, w^{op}), -k)$.

Hence, the composition $\phi = \pi\psi\pi_{op}^{-1}\colon S(L) \to S(L^{op})$ is an isomorphism, and we can give its explicit form.

$$\begin{tikzcd} S(L) \arrow[r, dashed, "\phi"] \arrow[d, "\pi"'] & S(L^{op}) \\ P^*(\mathbb{Z},\mathcal{X}^*,\mathcal{Y}^*) \arrow[r, "\psi"'] & P^*(\mathbb{Z},\mathcal{X}^*_{op},\mathcal{Y}^*_{op}) \arrow[u, "\pi_{op}"'] \end{tikzcd}$$

Since, by Remark 5.2.2, $\pi_{op}\colon S(L^{op}) \to P^*(\mathbb{Z},\mathcal{X}^*_{op},\mathcal{Y}^*_{op})$ is defined on the non-zero elements by $(i,w,j)\pi_{op} = ((i,w), j-i)$, then $\pi_{op}^{-1}\colon P^*(\mathbb{Z},\mathcal{X}^*_{op},\mathcal{Y}^*_{op}) \to S(L^{op})$ is defined on the non-zero elements by $((m,w),k)\pi_{op}^{-1} = (m,w,m+k)$. Therefore, for all non-zero $(i,w,j) \in S(L)$, we have

$$\begin{aligned}(i,w,j)\phi &= (i,w,j)\pi\psi\pi_{op}^{-1}\\ &= ((i,w),j-i)\psi\pi_{op}^{-1}\\ &= ((|w|-i+1,w^{op}),-(j-i))\pi_{op}^{-1}\\ &= ((|w|-i+1,w^{op}),i-j)\pi_{op}^{-1}\\ &= (|w|-i+1,w^{op},|w|-i+1+i-j)\\ &= (|w|-i+1,w^{op},|w|-j+1)\,.\end{aligned}$$

In Chapter 7, Section 7.3, we will see that, given a factorial language L containing all words over its alphabet up to length 3, the isomorphism $\phi\colon S(L) \to S(L^{op})$ obtained in the previous example is, along with the identity map on $S(L)$, the only isomorphism from $S(L)$ onto an inverse semigroup associated with a factorial language, up to a bijective correspondence between their respective alphabets.

To conclude, we present an example where the P^*-semigroups are isomorphic but the corresponding P-semigroups are not isomorphic. This time (cf. Example 5.3.1), although the groups involved are isomorphic (in fact, coinciding), the isomorphism cannot be written by means of a compatible pair.

The following easy fact will be useful:

Lemma 5.3.3. *If L and K are factorial languages over an alphabet Σ, then $L \cup K$ is a factorial language over Σ and $S(L \cup K) = S(L) \cup S(K)$. Further, if $s,t \in S(L \cup K)$, then $st \neq 0$ implies that $s,t,st \in S(L)$ or $s,t,st \in S(K)$.*

As established in the subsection of Section 3.2 concerning one-dimensional tiling semigroups, we denote by $\underline{s}$ the underlying word of a non-zero element s in the inverse semigroup associated with a factorial language.

Proof. That $L \cup K$ is factorial and $S(L \cup K) = S(L) \cup S(K)$ are obvious. Let $s,t \in S(L \cup K)$ be such that $st \neq 0$. Then $\underline{st}$ is a word in $L \cup K$. If $\underline{st} \in L$, then $\underline{s},\underline{t} \in L$ as L is factorial, and we have that $s,t,st \in S(L)$. Similarly, if $\underline{st} \in K$, then $s,t,st \in S(K)$. ☐

Example 5.3.4. Let L and K be two factorial languages over a fixed alphabet such that $L \cap (K \cup K^{op})$ contains no word of length two, and therefore no longer word. By the lemma above, both $S(L \cup K)$ and $S(L \cup K^{op})$ are inverse semigroups associated with factorial languages. We claim that they are isomorphic.

Define $\varphi\colon S(L \cup K) \to S(L \cup K^{op})$ on the non-zero elements by

$$(i,w,j)\varphi = \begin{cases}(i,w,j), & \text{if } w \in L\\ (|w|-i+1,w^{op},|w|-j+1), & \text{if } w \in K\,.\end{cases}$$

Since $w \in L \cap K$ if and only if $|w| = 1$, that is, w is a letter, then for every (i, w, j) with $w \in L \cap K$ we must have $i = j = 1$. It follows that

$$(|w| - i + 1, w^{op}, |w| - j + 1) = (1 - 1 + 1, w, 1 - 1 + 1) = (1, w, 1)\,,$$

and so φ is well-defined. In view of Lemma 5.3.3 and the previous example, we conclude that φ is an isomorphism.

Next, we show that, considering the P^*-representation $P_S^* = P^*(\mathbb{Z}, \mathcal{X}_S^*, \mathcal{Y}_S^*)$ of $S = S(L \cup K)$ and the P^*-representation $P_T^* = P^*(\mathbb{Z}, \mathcal{X}_T^*, \mathcal{Y}_T^*)$ of $T = S(L \cup K^{op})$ given by Theorem 5.1.2, it does not exist a compatible pair (α, Φ), with $\alpha\colon \mathbb{Z} \to \mathbb{Z}$ a group isomorphism and $\Phi\colon \mathcal{X}_S \to \mathcal{X}_T$ an order isomorphism, such that the isomorphism $\phi = \pi_S^{-1}\varphi\pi_T\colon P_S^* \to P_T^*$ is given by $((m, w), k)\phi = ((m, w)\Phi, k\alpha)$, for all $((m, w), k) \in P_S^*$. In fact, if $w \in L$ and $v \in K$ are two-letter words, then $(1, w, 2)$ and $(1, v, 2)$ are non-zero elements from S. On the one hand, the representation of $(1, w, 2)$ and $(1, v, 2)$ in the P^*-semigroup $P_S^* = (\mathbb{Z}, \mathcal{X}_S^*, \mathcal{Y}_S^*)$ is $((1, w), 1)$ and $((1, v), 1)$, respectively. On the other hand, the representation of the elements $(1, w, 2)\varphi = (1, w, 2)$ and $(1, v, 2)\varphi = (2 - 1 + 1, v^{op}, 2 - 2 + 1) = (2, v^{op}, 1)$ from T in the P^*-semigroup $P_T^* = (\mathbb{Z}, \mathcal{X}_T^*, \mathcal{Y}_T^*)$ is $((1, w), 1)$ and $((2, v^{op}), -1)$, respectively. Thus, we have

$$((1, w), 1)\phi = ((1, w), 1)\pi_S^{-1}\varphi\pi_T = (1, w, 2)\varphi\pi_T = (1, w, 2)\pi_T = ((1, w), 1)\,,$$

since $w \in L$, while

$$((1, v), 1)\phi = ((1, v), 1)\pi_S^{-1}\varphi\pi_T = (1, v, 2)\varphi\pi_T = (2, v^{op}, 1)\pi_T = ((2, v^{op}), -1)\,,$$

since $v \in K$. Therefore, if $\phi\colon P_S^* \to P_T^*$ was defined by $((m, w), k)\phi = ((m, w)\Phi, k\alpha)$, for all $((m, w), k) \in P_S^*$, then, in particular,

$$\begin{aligned} ((1, w), 1) &= ((1, w), 1)\phi = ((1, w)\Phi, 1\alpha) \text{ and} \\ ((2, v^{op}), -1) &= ((1, v), 1)\phi = ((1, v)\Phi, 1\alpha)\,. \end{aligned}$$

But then $1\alpha = 1$ and $1\alpha = -1$, which is impossible. Hence, there does not exist a compatible pair (α, Φ) such that the isomorphism $\phi\colon P_S^* \to P_T^*$ is given by $((m, w), k)\phi = ((m, w)\Phi, k\alpha)$, for all $((m, w), k) \in P_S^*$.

In Chapter 7, this example will provide an immediate counter-example within our investigations into isomorphic inverse semigroups associated with factorial languages. More precisely, it will show that we may have $S(L_1) \simeq S(L_2)$, with L_1 and L_2 factorial languages over Σ_1 and Σ_2, without $L_2 = L_1\varphi^*$ or $L_2 = L_1^{op}\varphi^*$, with $\varphi\colon \Sigma_1 \to \Sigma_2$ a bijection and $\varphi^*\colon L_1 \to L_2$ the map that naturally extends φ, in case L_1 and L_2 do not contain all words of length 3 over their alphabets.

Chapter 6

Presentation of hypercubic tilings

In this chapter, we investigate hypercubic tiling semigroups with respect to inverse semigroup presentations. Despite the philosophy adopted so far, of conducting our study within the more general framework of inverse semigroups associated with n-dimensional factorial languages, and then drawing the corresponding conclusions for n-dimensional hypercubic tilings, we shall do that only in dimension 1. The reason for this is that, with respect to this topic, working with n-dimensional factorial languages implies, for $n \geqslant 2$, an unjustified overload of technicalities.

Unlike all the previous topics studied in this thesis, one-dimensional and higher dimensional hypercubic tiling semigroups have, regarding presentations, largely contrasting behaviours. While, in Section 6.1, we will construct a presentation for the inverse semigroup associated with a (one-dimensional) factorial language, and subsequently investigate when the same is finitely presented, Section 6.2 is dedicated to proving that an n-dimensional hypercubic tiling semigroup with $n \geqslant 2$ is always infinitely presented, even within the class of strongly E^*-unitary inverse semigroups that admit a 0-restricted idempotent-pure pre-homomorphism into an abelian group with zero. In particular, we will see that one-dimensional periodic tiling semigroups are always finitely presented, as inverse semigroups, while n-dimensional periodic hypercubic tiling semigroups, with $n \geqslant 2$, need not be so; in fact, even such a semigroup may require an infinite number of defining relations beyond those that an n-dimensional tiling semigroup over the same alphabet containing all possible patterns already does.

6.1 A presentation for one-dimensional tilings

As mentioned, in this section we investigate a general way to give a presentation of the semigroup inverse $S(L)$ associated with a one-dimensional factorial language L over a finite alphabet Σ. We begin by studying some examples that motivate the course taken for the general case. Next, having found a presentation for $S(L)$, with L an arbitrary factorial language, we examine under which conditions the presentation is finite and, in fact, the semigroup finitely presented as an inverse semigroup. Finally, we examine two classes of families of languages with respect to the problem of finite presentability.

Some motivating examples

Let us then consider a few examples that help to get a better insight into the problem.

Example 6.1.1. Consider the factorial language $L = a^+$ over the alphabet $\{a\}$. Notice that L is the language of the tiling whose associated bi-infinite word is

$$\dots a\,a\,a\,a\,a\,a\,a\,a \dots$$

As shown by Lawson in [32], $S(L)$ is isomorphic to the free inverse monoid with one generator (note that, in this case, the semigroup may be regarded as not having a zero). Therefore, by [7, Remark 1.3],

$$S(L) = Inv\,\langle e, x \mid ee = e, ex = xe = x\rangle \ .$$

Example 6.1.2. Now let $\Sigma = \{a_1, \dots, a_n\}$ with $n \geqslant 2$ be a finite alphabet and consider $S(\Sigma^+)$, the free one-dimensional hypercubic tiling semigroup over Σ (cf. Example 4.2.7). In $S(\Sigma^+)$, the product of two non-zero elements is non-zero provided that the sequences match, as the resulting underlying word trivially belongs to the language.

From what we saw in Section 3.4, $S(\Sigma^+)$ is generated, as an inverse semigroup with zero, by its one-letter idempotents $\breve{a}_i$ with $i \in [n]$ and the two-letter elements $\grave{a}_i\acute{a}_j$ with $i, j \in [n]$.

Let K be the inverse subsemigroup of $S(\Sigma^+)$ generated by the two-letter elements $\grave{a}_i\acute{a}_j$ with $i, j \in [n]$. Since the product of two elements s and t from $S(\Sigma^+)$ is either 0 or an element whose underlying word has length greater or equal than the length of the underlying word of s and the underlying word of t, then $K = S(\Sigma^+) \setminus \{\breve{a}_1, \dots, \breve{a}_n\}$ and is in fact an ideal of $S(\Sigma^+)$.

In what follows, a slightly different representation — than the one given by Lawson's canonical form (cf. Section 3.3) — of the elements of $S(\Sigma^+)$ that belong to K will be advantageous. Let $\Sigma^{\geqslant 2}$ denote the set of words over Σ with at least two letters. Consider the map $\tau\colon \Sigma^{\geqslant 2} \to K$ defined by

$$(a_{i_1} \dots a_{i_m})\tau = \grave{a}_{i_1} \dots \acute{a}_{i_m}\ ,$$

for all $a_{i_1} \dots a_{i_m} \in \Sigma^{\geqslant 2}$. Note that

$$(a_{i_1} \dots a_{i_m})\tau = (a_{i_1}a_{i_2})\tau\,(a_{i_2}a_{i_3})\tau\ \dots (a_{i_{m-1}}a_{i_m})\tau\ .$$

Now, if $a_{i_1} \dots a_{i_m}$, with $m \geqslant 2$, is the underlying word of $s \in K$ with in-letter a_{i_j} and out-letter a_{i_k}, then

$$s = \begin{cases} (a_{i_1} \dots a_{i_m})\tau & \text{if } j = 1 \text{ and } k = m \\ (a_{i_1} \dots a_{i_m})\tau\,(a_{i_k} \dots a_{i_m})\tau^{-1} & \text{if } j = 1 \text{ and } k \neq m \\ (a_{i_1} \dots a_{i_j})\tau^{-1}\,(a_{i_1} \dots a_{i_m})\tau & \text{if } j \neq 1 \text{ and } k = m \\ (a_{i_1} \dots a_{i_j})\tau^{-1}\,(a_{i_1} \dots a_{i_m})\tau\,(a_{i_k} \dots a_{i_m})\tau^{-1} & \text{if } j \neq 1 \text{ and } k \neq m\,. \end{cases}$$

It is important to note that this representation is unique.

In order to obtain a presentation for $S(\Sigma^+)$, we begin by computing the products of the generators of $S(\Sigma^+)$. They are:

$$
\begin{array}{rl}
(1) & \breve{a}_i\,\breve{a}_i = \breve{a}_i \\
(2) & \breve{a}_i\,\breve{a}_j = 0 \text{ for } i \neq j \\
(3) & \breve{a}_i\,\grave{a}_i\acute{a}_j = \grave{a}_i\acute{a}_j \\
(4) & \grave{a}_i\acute{a}_j\,\breve{a}_j = \grave{a}_i\acute{a}_j \\
(5) & \breve{a}_k\,\grave{a}_i\acute{a}_j = 0 \text{ for } k \neq i \\
(6) & \grave{a}_i\acute{a}_j\,\breve{a}_k = 0 \text{ for } k \neq j \\
(7) & \grave{a}_i\acute{a}_j\,\grave{a}_k\acute{a}_l = \grave{a}_i a_j \acute{a}_l \text{ for } k = j \\
(8) & \grave{a}_i\acute{a}_j\,\grave{a}_k\acute{a}_l = 0 \text{ for } k \neq j \\
(9) & \grave{a}_i\acute{a}_j\,\grave{a}_i\acute{a}_j^{-1} = \breve{a}_i a_j \\
(10) & \grave{a}_i\acute{a}_j\,\grave{a}_k\acute{a}_l^{-1} = 0 \text{ for } i \neq k \text{ or } j \neq l \\
(11) & \grave{a}_i\acute{a}_j^{-1}\,\grave{a}_i\acute{a}_j = a_i\breve{a}_j \\
(12) & \grave{a}_i\acute{a}_j^{-1}\,\grave{a}_k\acute{a}_l = 0 \text{ for } i \neq k \text{ or } j \neq l\,.
\end{array}
$$

From the fact that $S(\Sigma^+)$ is an inverse semigroup with zero, some of these products need not be considered as defining relations:

- since, for $k \neq i$,
$$\breve{a}_k\,\grave{a}_i\acute{a}_j \overset{(3)}{=} \breve{a}_k\,\breve{a}_i\,\grave{a}_i\acute{a}_j \overset{(2)}{=} 0\,\grave{a}_i\acute{a}_j = 0\,,$$
(5) can be omitted;

- similarly, since, for $k \neq j$,
$$\grave{a}_i\acute{a}_j\,\breve{a}_k \overset{(4)}{=} \grave{a}_i\acute{a}_j\,\breve{a}_j\,\breve{a}_k \overset{(2)}{=} \grave{a}_i\acute{a}_j\,0 = 0\,,$$
(6) can be omitted as well;

- since $\grave{a}_i a_j \acute{a}_l$ is written as $\grave{a}_i\acute{a}_j\,\grave{a}_j\acute{a}_l = \grave{a}_i\acute{a}_j\,\grave{a}_k\acute{a}_l$ (with $j = k$) in terms of the generators of S, (7) can be omitted;

- since, for $k \neq j$,
$$\grave{a}_i\acute{a}_j\,\grave{a}_k\acute{a}_l \overset{(3)\text{ and }(4)}{=} (\grave{a}_i\acute{a}_j\,\breve{a}_j)\,(\breve{a}_k\,\grave{a}_k\acute{a}_l) = \grave{a}_i\acute{a}_j\,\breve{a}_j\,\breve{a}_k\,\grave{a}_k\acute{a}_l \overset{(2)}{=} \grave{a}_i\acute{a}_j\,0\,\grave{a}_k\acute{a}_l = 0\,,$$
(8) can also be omitted;

- since the idempotent $\breve{a}_i a_j$ (respectively, $a_i\breve{a}_j$) is written in terms of the generators of S (and their inverses) as $\grave{a}_i\acute{a}_j\,\grave{a}_i\acute{a}_j^{-1}$ (respectively, $\grave{a}_i\acute{a}_j^{-1}\,\grave{a}_i\acute{a}_j$), (9) (respectively, (11)) can be omitted.

In addition, both (10) and (12) are partly a consequence of other products:

- for $j \neq l$,
$$\grave{a}_i\acute{a}_j\,\grave{a}_k\acute{a}_l^{-1} \overset{(4)}{=} \grave{a}_i\acute{a}_j\,\breve{a}_j\,(\grave{a}_k\acute{a}_l\,\breve{a}_l)^{-1} \overset{(1)}{=} \grave{a}_i\acute{a}_j\,\breve{a}_j\,\breve{a}_l\,\grave{a}_k\acute{a}_l^{-1} \overset{(2)}{=} 0\,,$$
and so we may consider only
$$\text{(10')} \qquad \grave{a}_i\acute{a}_j\,\grave{a}_k\acute{a}_l^{-1} = 0 \text{ for } i \neq k\,;$$

- likewise, since, for $i \neq k$,
$$\grave{a}_i\acute{a}_j^{-1}\,\grave{a}_k\acute{a}_l \overset{(3)}{=} (\breve{a}_i\,\grave{a}_i\acute{a}_j)^{-1}\,\breve{a}_k\,\grave{a}_k\acute{a}_l \overset{(1)}{=} \grave{a}_i\acute{a}_j^{-1}\,\breve{a}_i\,\breve{a}_k\,\grave{a}_k\acute{a}_l \overset{(2)}{=} 0\,,$$
we may consider
$$\text{(12')} \qquad \grave{a}_i\acute{a}_j^{-1}\,\grave{a}_k\acute{a}_l = 0 \text{ for } j \neq l\,.$$

In fact, we may even consider instead

$$\text{(10'')} \qquad \grave{a}_i\acute{a}_j\,\grave{a}_k\acute{a}_j^{-1} = 0 \text{ for } i \neq k$$
$$\text{(12'')} \qquad \grave{a}_i\acute{a}_j^{-1}\,\grave{a}_i\acute{a}_l = 0 \text{ for } j \neq l\,.$$

This leads us to consider the following set of relations:

$$
\begin{array}{rl}
\text{(i)} & \breve{a}_i\,\breve{a}_i = \breve{a}_i \\
\text{(ii)} & \breve{a}_i\,\breve{a}_j = 0 \text{ for } i \neq j \\
\text{(iii)} & \breve{a}_i\,\grave{a}_i\acute{a}_j = \grave{a}_i\acute{a}_j \\
\text{(iv)} & \grave{a}_i\acute{a}_j\,\breve{a}_j = \grave{a}_i\acute{a}_j \\
\text{(v)} & \grave{a}_i\acute{a}_j\,\grave{a}_k\acute{a}_j^{-1} = 0 \text{ for } i \neq k \\
\text{(vi)} & \grave{a}_i\acute{a}_j^{-1}\,\grave{a}_i\acute{a}_l = 0 \text{ for } j \neq l\,.
\end{array}
$$

By reproducing our arguments above, it is easy to check that

Lemma 6.1.3. *The relations (i)–(vi) imply the relations*

(vii) $\breve{a}_k\,\grave{a}_i\acute{a}_j = 0$, *for all* $k \neq i$, *and* $\grave{a}_i\acute{a}_j\,\breve{a}_l = 0$, *for all* $l \neq j$;

(viii) $\grave{a}_i\acute{a}_j^{-1}\,\grave{a}_k\acute{a}_l = 0$ *and* $\grave{a}_i\acute{a}_j\,\grave{a}_k\acute{a}_l^{-1} = 0$, *for all* $(i,j) \neq (k,l)$.

Consider the McAlister semigroup on n^2 generators

$$M_{n^2} = Inv_0\left\langle x_{ij}\ (i,j \in [n]) \mid x_{ij}x_{kl}^{-1} = 0 = x_{ij}^{-1}x_{kl} \text{ for } (i,j) \neq (k,l)\right\rangle\,.$$

Since $S(\Sigma^+)$, and in particular K, the inverse subsemigroup of $S(\Sigma^+)$ generated by the two-letter generators, satisfy the identities $\grave{a}_i\acute{a}_j\,\grave{a}_k\acute{a}_l^{-1} = 0$ and $\grave{a}_i\acute{a}_j^{-1}\,\grave{a}_k\acute{a}_l = 0$ for $(i,j) \neq (k,l)$, we have that K is a homomorphic image of M_{n^2}. In fact,

Lemma 6.1.4. *Let I be the ideal of M_{n^2} generated by the set $\{x_{ij}x_{kl} \colon j \neq k\}$. Then $K \simeq M_{n^2}/I$.*

Proof. Recall from Section 3.3 that the non-zero elements of M_{n^2} can be written in the form $u^{-1}vw^{-1}$, with $u, v, w \in \{x_i \colon i \in [n]\}^*$, $v \neq 1$, u a prefix of v and w a suffix of v. Thus, the non-zero elements of I are those in which the word v contains at least one factor of the form $x_{ij}x_{kl}$ with $j \neq k$ and so $M_{n^2} \setminus I$ consists of the elements in which the word v contains no factor of the form $x_{ij}x_{kl}$ with $j \neq k$.

Consider the epimorphism $\varphi \colon M_{n^2} \to K$ which extends the map that sends each generator x_{ij} of M_{n^2} to the generator $\grave{a}_i\acute{a}_j$ of K. Then, in view of the representation of the non-zero elements of K in terms of the map τ, we have that $(u^{-1}vw^{-1})\varphi = u\tau^{-1}\, v\tau\, w\tau^{-1}$ if v is a word of the form $x_{i_1i_2}x_{i_2i_3}\dots x_{i_{m-1}i_m}$ and $(u^{-1}vw^{-1})\varphi = 0$ otherwise, that is, if $u^{-1}vw^{-1} \in I$. Therefore, by the uniqueness of the representation in K, we have $(u^{-1}vw^{-1})\varphi = (r^{-1}st^{-1})\varphi$ if and only if $u^{-1}vw^{-1} = r^{-1}st^{-1}$ or $u^{-1}vw^{-1}, r^{-1}st^{-1} \in I$. But then $\ker\varphi = \ker\rho_I$, and so the epimorphism $\bar{\varphi}$ that factors φ through $\rho_I^\natural$ is in fact an isomorphism.

$$\begin{tikzcd}
M_{n^2} \arrow[r, two heads, "\varphi"] \arrow[d, "\rho_I^\natural"'] & K \\
M_{n^2}/I \arrow[ur, dashed, "\bar{\varphi}"'] &
\end{tikzcd}$$

Hence, $K \simeq M_{n^2}/I$. □

By Proposition 1.2.13, the following is immediate:

Corollary 6.1.5.

$$\begin{aligned} K = Inv_0 \,\langle \grave{a}_i\acute{a}_j\ (i,j \in [n]) \mid & \grave{a}_i\acute{a}_j\, \grave{a}_k\acute{a}_l^{-1} = 0 = \grave{a}_i\acute{a}_j^{-1}\, \grave{a}_k\acute{a}_l \textit{ for } (i,j) \neq (k,l), \\ & \grave{a}_i\acute{a}_j\, \grave{a}_k\acute{a}_l = 0 \textit{ for } j \neq k \,\rangle\ . \end{aligned}$$

We can now produce a presentation for $S(\Sigma^+)$. Notice that the semigroup

$$\begin{aligned} S = Inv_0 \big\langle \breve{a}_i\ (i \in [n]), \grave{a}_i\acute{a}_j\ (i,j \in [n]) \,|\, & \breve{a}_i\, \breve{a}_i = \breve{a}_i\ (i \in [n]),\ \breve{a}_i\, \breve{a}_j = 0\ (i,j \in [n] \text{ with } i \neq j), \\ & \breve{a}_i\, \grave{a}_i\acute{a}_j = \grave{a}_i\acute{a}_j\ (i,j \in [n]),\ \grave{a}_i\acute{a}_j\, \breve{a}_j = \grave{a}_i\acute{a}_j\ (i,j \in [n]), \\ & \breve{a}_k\, \grave{a}_i\acute{a}_j = 0\ (i,j \in [n] \text{ with } i \neq k),\ \grave{a}_i\acute{a}_j\, \breve{a}_k = 0\ (i,j \in [n] \text{ with } j \neq k), \\ & \grave{a}_i\acute{a}_j\, \grave{a}_k\acute{a}_l^{-1} = 0 = \grave{a}_i\acute{a}_j^{-1}\, \grave{a}_k\acute{a}_l\ (i,j,k,l \in [n] \text{ with } (i,j) \neq (k,l)), \\ & \grave{a}_i\acute{a}_j\, \grave{a}_k\acute{a}_l = 0\ (i,j,k,l \in [n] \text{ with } j \neq k) \big\rangle\ . \end{aligned}$$

is the disjoint union of $\{\breve{a}_1, \dots, \breve{a}_n\}$ and its ideal W generated by $\{\grave{a}_i\acute{a}_j \colon i,j \in [n]\}$. Thus, the mapping $\psi \colon S \to S(\Sigma^+)$ that sends each generator $\breve{a}_i$ to a generator $\breve{a}_i$ and each generator $\grave{a}_i\acute{a}_j$ to a generator $\grave{a}_i\acute{a}_j$ extends to a homomorphism and we know that its restriction to W

is bijective, since, from what we have seen before, W and K are isomorphic. Moreover, the restriction of ψ to the one-letter generators $\breve{a}_i$ is evidently bijective as well. Consequently, $S \simeq S(\Sigma^+)$ and so

$$\begin{aligned} S(\Sigma^+) = Inv_0 \langle \breve{a}_i \ (i \in [n]), \grave{a}_i\acute{a}_j \ (i,j \in [n]) \mid\ & \breve{a}_i\, \breve{a}_i = \breve{a}_i \ (i \in [n]),\ \breve{a}_i\, \breve{a}_j = 0 \ (i,j \in [n] \text{ with } i \neq j), \\ & \breve{a}_i\, \grave{a}_i\acute{a}_j = \grave{a}_i\acute{a}_j \ (i,j \in [n]),\ \grave{a}_i\acute{a}_j\, \breve{a}_j = \grave{a}_i\acute{a}_j \ (i,j \in [n]), \\ & \breve{a}_k\, \grave{a}_i\acute{a}_j = 0 \ (i,j \in [n] \text{ with } i \neq k),\ \grave{a}_i\acute{a}_j\, \breve{a}_k = 0 \ (i,j \in [n] \text{ with } j \neq k), \\ & \grave{a}_i\acute{a}_j\, \grave{a}_k\acute{a}_l^{-1} = 0 = \grave{a}_i\acute{a}_j^{-1}\, \grave{a}_k\acute{a}_l \ (i,j,k,l \in [n] \text{ with } (i,j) \neq (k,l)), \\ & \grave{a}_i\acute{a}_j\, \grave{a}_k\acute{a}_l = 0 \ (i,j,k,l \in [n] \text{ with } j \neq k) \rangle \ . \end{aligned}$$

Finally, since by Lemma 6.1.3 the relations in this presentation are a consequence of a smaller set of relations, we have

Theorem 6.1.6. *Let* $\Sigma = \{a_1, \ldots, a_n\}$. *Then*

$$\begin{aligned} S(\Sigma^+) = Inv_0 \Big\langle \breve{a}_i \ (i \in [n]), \grave{a}_i\acute{a}_j \ (i,j \in [n]) \mid\ & \breve{a}_i\, \breve{a}_i = \breve{a}_i \ (i \in [n]),\ \breve{a}_i\, \breve{a}_j = 0 \ (i,j \in [n] \textit{ with } i \neq j), \\ & \breve{a}_i\, \grave{a}_i\acute{a}_j = \grave{a}_i\acute{a}_j \ (i,j \in [n]),\ \grave{a}_i\acute{a}_j\, \breve{a}_j = \grave{a}_i\acute{a}_j \ (i,j \in [n]), \\ & \grave{a}_i\acute{a}_j\, \grave{a}_k\acute{a}_j^{-1} = 0 \ (i,j,k \in [n] \textit{ with } i \neq k), \\ & \grave{a}_i\acute{a}_j^{-1}\, \grave{a}_i\acute{a}_k = 0 \ (i,j,l \in [n] \textit{ with } j \neq l) \Big\rangle \ . \end{aligned}$$

A final example shows how the presentation for $S(\Sigma^+)$ can be used to find a presentation for the semigroup associated with a particular factorial language.

Example 6.1.7. Consider the language $L = L(\mathcal{T})$, where $\mathcal{T}$ is the tiling

$$\cdots a\, a\, a\, a\, a\, a\, b\, b\, b\, b\, b\, b \cdots$$

We see that L contains every word over the alphabet $\Sigma = \{a, b\}$ which does not have the word ba as a factor. It follows from Proposition 4.2.8 that $S(L) = S(\Sigma^+)/I(L)$, where $I(L)$ is the ideal of $S(\Sigma^+)$ consisting of 0 together with all elements whose underlying word contains the factor ba. Since $I(L)$ is generated, as an ideal in an inverse semigroup with zero, by $\grave{b}\acute{a}$, in view of Proposition 1.2.13 a presentation for the associated semigroup $S(L)$ is

$$\begin{aligned} Inv_0 \langle\, \breve{a}, \breve{b}, \grave{a}\acute{a}, \grave{a}\acute{b}, \grave{b}\acute{a}, \grave{b}\acute{b} \mid\ & \breve{a}\,\breve{a} = \breve{a}, \breve{b}\,\breve{b} = \breve{b}, \breve{a}\,\breve{b} = \breve{b}\,\breve{a} = 0, \\ & \breve{a}\,\grave{a}\acute{a} = \grave{a}\acute{a}, \breve{a}\,\grave{a}\acute{b} = \grave{a}\acute{b}, \breve{b}\,\grave{b}\acute{b} = \grave{b}\acute{b}, \grave{a}\acute{a}\,\breve{a} = \grave{a}\acute{a}, \grave{a}\acute{b}\,\breve{b} = \grave{a}\acute{b}, \grave{b}\acute{b}\,\breve{b} = \grave{b}\acute{b}, \\ & \grave{a}\acute{b}\,\grave{b}\acute{b}^{-1} = \grave{b}\acute{b}\,\grave{a}\acute{b}^{-1} = \grave{a}\acute{a}^{-1}\,\grave{a}\acute{b} = \grave{a}\acute{b}^{-1}\,\grave{a}\acute{a} = 0, \\ & \grave{b}\acute{a} = 0 \,\rangle \ . \end{aligned}$$

Notice that, since $\grave{b}\acute{a} = 0$, there is no need to consider any other relation involving this generator.

The general case

We now turn to the general case. Thus, let L be an arbitrary factorial language over the alphabet $\Sigma = \{a_1, \ldots, a_n\}$. To avoid trivialities, we shall assume that every letter of Σ appears in some word of L. Since L is factorial, this is equivalent to having $a \in L$, for each $a \in \Sigma$. By definition of $S(L)$, the product of two non-zero elements is non-zero in $S(L)$ not only if the sequences match, but also if the underlying word of the resulting sequence belongs to L. Also recall, from Section 3.4, that $S(L)$ is generated, as an inverse semigroup with zero, by its one-letter idempotents $\check{a}$, with $a \in \Sigma$ (or equivalently, $a \in L$), together with its two-letter elements of the form $\grave{a}_i\acute{a}_j$, with $a_i a_j \in L$.

Our strategy for computing a presentation for $S(L)$ is to generalize the argument used in Example 6.1.7, since by Proposition 4.2.8 we have $S(L) \simeq S(\Sigma^+)/I(L)$, with

$$I(L) = \big\{s \in S(\Sigma^+)\colon \text{the underlying word of } s \text{ does not belong to } L\big\} \cup \{0\} \ .$$

Consider the alphabet

$$A = \{\check{a}_i \colon i \in [n]\} \cup \{\grave{a}_i\acute{a}_j \colon i, j \in [n]\} \ .$$

In Example 6.1.2, we considered the map $\tau\colon \Sigma^{\geqslant 2} \to K$ defined by

$$(a_{i_1} \ldots a_{i_m})\tau = \grave{a}_{i_1} \ldots \acute{a}_{i_m} \ ,$$

for all words $a_{i_1} \ldots a_{i_m}$ over Σ with at least two letters. Also recall that

$$(a_{i_1} \ldots a_{i_m})\tau = (a_{i_1}a_{i_2})\tau \, (a_{i_2}a_{i_3})\tau \ \ldots (a_{i_{m-1}}a_{i_m})\tau \, .$$

With some abuse of notation, we will now denote by τ the mapping $\tau\colon \Sigma^{\geqslant 2} \to A^+$ defined by

$$(a_{i_1} \ldots a_{i_m})\, \tau = \grave{a}_{i_1}\acute{a}_{i_2} \, \grave{a}_{i_2}\acute{a}_{i_3} \ \ldots \ \grave{a}_{i_{m-1}}\acute{a}_{i_m} \, .$$

for all $a_{i_1} \ldots a_{i_m}$ with $m \geqslant 2$. Thus, for every word $a_{i_1} \ldots a_{i_m}$ from Σ^+ with at least two letters, $(a_{i_1} \ldots a_{i_m})\tau$ represents the element $\grave{a}_{i_1} \ldots \acute{a}_{i_m} \in S(\Sigma^+)$ in terms of the generators from A.

Given that, from Theorem 6.1.6, we have

$$\begin{aligned} S(\Sigma^+) = Inv_0 \, \langle A \,|\, & \check{a}_i \, \check{a}_i = \check{a}_i \ (i \in [n]), \ \check{a}_i \, \check{a}_j = 0 \ (i, j \in [n] \text{ with } i \neq j), \\ & \check{a}_i \, \grave{a}_i\acute{a}_j = \grave{a}_i\acute{a}_j, \ \grave{a}_i\acute{a}_j \, \check{a}_j = \grave{a}_i\acute{a}_j \ (i, j \in [n]), \\ & \grave{a}_i\acute{a}_j \, \grave{a}_k\acute{a}_j{}^{-1} = 0 \ (i, j, k \in [n] \text{ with } i \neq k), \\ & \grave{a}_i\acute{a}_j{}^{-1} \, \grave{a}_i\acute{a}_l = 0 \ (i, j, l \in [n] \text{ with } j \neq l) \ \rangle \ , \end{aligned}$$

with A as above, we conclude the following

Theorem 6.1.8. *Let L be a factorial language over Σ. Let*

$$A = \{\breve{a} \colon a \in \Sigma\} \cup \{\grave{a}_i\acute{a}_j \colon a_i, a_j \in \Sigma\} \tag{6.1}$$

and

$$\begin{aligned} R_S = &\{\breve{a}\,\breve{a} = \breve{a} \colon a \in \Sigma\} \cup \{\breve{a}_i\,\breve{a}_j = 0 \colon a_i, a_j \in \Sigma, a_i \neq a_j\} \\ &\cup \{\breve{a}_i\,\grave{a}_i\acute{a}_j = \grave{a}_i\acute{a}_j,\ \grave{a}_i\acute{a}_j\,\breve{a}_j = \grave{a}_i\acute{a}_j \colon a_ia_j \in L\} \\ &\cup \left\{\grave{a}_i\acute{a}_j\,\grave{a}_k\acute{a}_j{}^{-1} = 0 \colon a_ia_j, a_ka_j \in L \text{ with } a_i \neq a_k\right\} \\ &\cup \left\{\grave{a}_i\acute{a}_j{}^{-1}\,\grave{a}_i\acute{a}_l = 0 \colon a_ia_j, a_ia_l \in L \text{ with } a_j \neq a_l\right\} . \end{aligned} \tag{6.2}$$

Then

$$S(L) = Inv_0 \langle A \mid u = v \in R_S, w\tau = 0\ (w \notin L)\rangle\ .$$

In order to efficiently discuss the finite or infinite presentability of $S(L)$, we first improve the presentation obtained. First, and again as in Example 6.1.7, we note that if a two-letter word a_ia_j does not belong to L, then $\grave{a}_i\acute{a}_j = 0$ is a relation of the form $w\tau = 0$, and so nothing is lost by not considering the relations in R_S involving this generator. The major improvement, however, will be on the number of relations used to impose that an element is zero in $S(L)$ if its underlying word is not in L. The most compact way to do so is, of course, to consider a minimal set of generators of $\Sigma^* \setminus L$. Recall from Remark 1.3.6 that, because L is factorial, $\Sigma^* \setminus L$ is an ideal of Σ^* and the set $M(L)$ of minimal forbidden words of L,

$$M(L) = \{w \in \Sigma^* \setminus L \colon w = axb, \text{ with } a, b \in \Sigma,\ x \in \Sigma^* \text{ and } ax, xb \in L\}\ ,$$

is the unique minimal set of generators of $\Sigma^* \setminus L$. Hence,

Theorem 6.1.9. *Let L be a factorial language over Σ. Then*

$$S(L) = Inv_0 \langle A \mid u = v \in R_S, w\tau = 0 \in R_L\rangle\ ,$$

where A is defined by (6.1), R_S is defined by (6.2), and

$$R_L = \{w\tau = 0 \colon w \in M(L)\}\ . \tag{6.3}$$

From a "philosophical" point of view, the relations of a presentation for $S(L)$ can be regarded as being divided into two sets: those that are a consequence of the "context-blind" part of the operation of $S(L)$, and are therefore universal for every semigroup constructed in this manner from a factorial language over Σ, and the relations that define which are the forbidden underlying words, if any, in that particular language. These sets have been denoted by R_S and R_L, respectively.

As expected, Theorem 6.1.9 yields, for the languages in Examples 6.1.2 and 6.1.7, precisely the presentation there obtained. Also note that the set of minimal forbidden words of those languages is, respectively, $M(L) = \emptyset$ and $M(L) = \{ba\}$.

Finiteness of the presentation

We now turn to the question of determining when the inverse semigroup associated with an arbitrary one-dimensional factorial language L is finitely presented (as an inverse semigroup, evidently).

By Theorem 6.1.6, we have that $S(\Sigma^+)$ is finitely presented as an inverse semigroup. In fact, if $|\Sigma| = n$, then $|A| = n + n^2$ and the number of relations in the presentation is

$$n + n(n-1) + 2n^2 + 2n^2(n-1) = n^2 + 2n^3\,.$$

Therefore, since $S(L) \simeq S(\Sigma^+)/I(L)$, it follows from Corollary 1.2.12 that $S(L)$ is finitely presented if and only if $I(L)$ is finitely generated, or, equivalently, $M(L)$ is finite. Thus,

Theorem 6.1.10. *Let L be a factorial language over a finite alphabet Σ. Then $S(L)$ is finitely presented if and only if the set $M(L)$ of minimal forbidden words of L is finite.*

In particular, we have that a one-dimensional tiling semigroup is finitely presented if and only if the tiling language has a finite number of minimal forbidden words.

Notice that, if $S(L)$ is finitely presented, then $M(L) = \{w_1, \ldots, w_m\}$, for some $w_1, \ldots, w_m \in \Sigma^+$. Since $\Sigma^* \setminus L = (\Sigma^* w_1 \Sigma^*) \cup \ldots \cup (\Sigma^* w_m \Sigma^*)$, it follows that $\Sigma^* \setminus L$ is a regular language, and therefore so is L. The converse, however, is not true, for it is possible for a regular factorial language not to have finitely presented associated inverse semigroup.

Example 6.1.11. Let L be the language recognized by the automaton

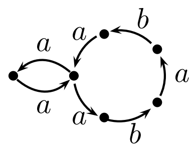


Figure 6.1: Regular factorial language with infinitely presented inverse semigroup

in which all states are both initial and final. Then L is regular (since it is recognized by a finite automaton) but $S(L)$ is not finitely presented. Indeed, for all odd $n \geqslant 3$, the word ba^nb is a minimal forbidden word of L, since $ba^nb \notin L$ while ba^n, $a^nb \in L$.

For the remainder of this section, we examine two families of regular factorial languages — periodic and ultimately periodic languages — with the objective of characterizing when is the inverse semigroup associated with a language from the family finitely presented.

Periodic tiling languages

Recall from Chapter 1 that a language L is said to be *periodic* if it consists of all factors of a bi-infinite word of the form $\ldots uuu \ldots$; that is, if $L = F(u^*)$, for some $u \in \Sigma^+$. Also recall that, for a factorial language L, to check that the proper factors of a word $w \in \Sigma^*$ belong to

L it suffices to show that its two factors of length $|w|-1$ belong to L. For this reason, it is not hard to see that a case when $\Sigma^* \setminus L$ is finitely generated occurs when L is periodic.

Recall that the words $u, v \in \Sigma^*$ are *conjugate* if there exist $x, y \in \Sigma^*$ such that $u = xy$ and $v = yx$. The following fact will be useful.

Lemma 6.1.12. *Let $u \in \Sigma^+$. If $v \in F(u^*)$ is such that $|u| = |v|$, then u and v are conjugate.*

Proof. The result holds trivially if $v = u$. If not, then v is a proper factor of uu, since $|v| = |u|$. Therefore $uu = xvy$, for some $x, y \in \Sigma^+$. As $|x|, |y| \leqslant |u|$, then x is a prefix of u and y is a suffix of u and since $|y| + |x| = |u|$, it follows that $xy = u$. Then $xyxy = uu = xvy$ implies that $v = yx$. By definition, u and v are conjugate. $\square$

In fact, it is easy to see that the converse is also true: if u and v are conjugate, then necessarily $|u| = |v|$ and $v \in F(u^*)$.

Proposition 6.1.13. *Let L be a periodic language over a finite alphabet. Then $\Sigma^* \setminus L$ is finitely generated.*

Proof. Let $u \in \Sigma^+$ be such that $L = F(u^*)$ and let $m = |u|$. We will prove that $\Sigma^* \setminus L$ is generated by the words $w \in \Sigma^* \setminus L$ such that $|w| \leqslant m$.

First we prove that a word of length $m+1$ whose prefix and suffix of length m belong to L must itself belong to L. Thus, let $z = a_1 \dots a_m a_{m+1} \in \Sigma^*$, with $a_1 \dots a_m$, $a_2 \dots a_{m+1} \in L$. Note that $z \in L$ if and only if $a_{m+1} = a_1$.

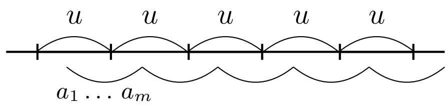
Now, $z = a_1 v = v' a_{m+1}$ with $v, v' \in F(u^*)$ and $|v| = |v'| = m$, so from the previous lemma, applied to u and v as well as to u and v', both v and v' are conjugates of u. Then there exist $x, y, x', y' \in \Sigma^*$ such that $u = xy = x'y'$ and $v = yx$ and $v' = y'x'$. Hence $z = a_1 yx = y'x' a_{m+1}$. Let χ be the epimorphism from Σ^* onto the free commutative monoid generated by Σ that satisfies $a\chi = a$, for all $a \in \Sigma$. Since χ is a homomorphism and $\Sigma^* / \ker \chi$ is commutative and cancellative, we have

$$\begin{aligned} z\chi = (a_1 yx)\chi = (y'x'a_{m+1})\chi &\Rightarrow a_1\chi\, y\chi\, x\chi = y'\chi\, x'\chi\, a_{m+1}\chi \\ &\Rightarrow a_1\chi\, x\chi\, y\chi = a_{m+1}\chi\, x'\chi\, y'\chi \\ &\Rightarrow a_1\chi\, (xy)\chi = a_{m+1}\chi\, (x'y')\chi \\ &\Rightarrow a_1\chi\, u\chi = a_{m+1}\chi\, u\chi \\ &\Rightarrow a_1\chi = a_{m+1}\chi \\ &\Rightarrow a_1 = a_{m+1}, \end{aligned}$$

as $a_1, a_{m+1} \in \Sigma$. Therefore, $z = a_1 \dots a_m a_1 \in L$.

Let $w = a_1 \dots a_m a_{m+1} \dots a_{m+p} \in \Sigma^* \setminus L$, with $p \geqslant 2$. For w to belong to a minimal set of generators of $\Sigma^* \setminus L$, we may assume, as we have noticed (cf. the paragraph after

Theorem 6.1.8), that $w \in M(L)$, that is, all proper factors of w belong to L. Consider the factor $a_1 \dots a_m$ of w. Since $a_1 \dots a_m \in L$ and $|a_1 \dots a_m| = m$,



we have $L = F((a_1 \dots a_m)^*)$. Then $a_{m+1} = a_1$, as $a_1 \dots a_m a_{m+1} \in L$, and, similarly, $a_{m+k} = a_r$ where r is the remainder of $m + k$ divided by m, for all $k \geqslant 1$. Thus $w \in L$, which is a contradiction.

Hence there are no words of length greater than m in a minimal set of generators of $\Sigma^* \setminus L$.

Finally, since Σ is finite, there exist only finitely many words of length bounded by m, and so $\Sigma^* \setminus L$ is finitely generated. □

As a consequence,

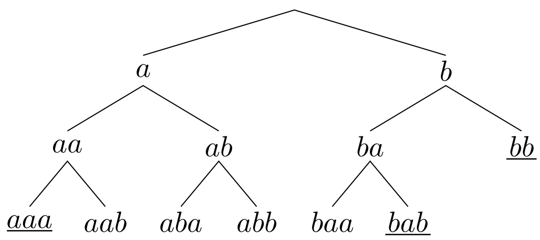
Theorem 6.1.14. *Let L be the language of a periodic (one-dimensional) tiling over a finite alphabet. Then $S(L)$ is finitely presented as an inverse semigroup.*

Notice that, from the proof of Proposition 6.1.13, to find a minimal generating set for $\Sigma^* \setminus L$ when $L = F(u^*)$ is a periodic language, we just have to check which words over Σ^* of length less or equal than $|u|$ are minimal forbidden factors of uu.

Example 6.1.15. Consider the periodic language $L = F((aba)^*)$.

$$\dots aba\, aba\, aba\, aba\, aba \dots$$

By inspection, we conclude that the ideal $\Sigma^* \setminus L$ is generated by $M(L) = \{b^2, a^3, bab\}$.



Two-way ultimately periodic languages

The converse of Theorem 6.1.14, of course, is not true, as shown for instance by Example 6.1.7: the language $L = F(a^*b^*)$ is not periodic — it is ultimately periodic (cf. Figure 6.2) — and yet $M(L) = \{ba\}$ is finite, so that $S(L)$ is finitely presented.

The next example shows that the inverse semigroup associated with an ultimately periodic language can also be infinitely presented:

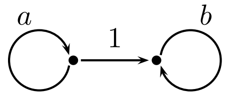


Figure 6.2: Ultimately periodic language with finitely presented semigroup

Example 6.1.16. Consider the ultimately periodic language $L = F((ab)^*bab(ba)^*)$ (cf. Figure 6.3). Then, for each $n \geqslant 1$, the word $bb(ab)^n abb$ is a minimal forbidden word for L, since $bbabb$ is the only word in L with bb as both prefix and suffix, but both $bb(ab)^n ab = bb(ab)^{n+1}$ and $b(ab)^n abb = (ba)^{n+1}bb$ belong to L. Therefore, $M(L)$ is infinite and so $S(L)$ is infinitely presented.

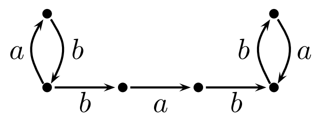


Figure 6.3: Ultimately periodic language with infinitely presented semigroup

To establish the characterization of finitely presented inverse semigroups associated with ultimately periodic languages, we begin by recalling two results from [50].

Proposition 6.1.17 ([50], Proposition 1.3.2)**.** *Two words $x, y \in \Sigma^+$ commute if and only if x and y are powers of the same word.*

Proposition 6.1.18 (Fine and Wilf, see [50], Proposition 1.3.5)**.** *Let $u, v \in \Sigma^*$, $p = |u|$, $q = |v|$, and $d = \gcd(p, q)$. If two powers u^m and v^n have a common prefix of length at least equal to $p + q - d$, then u and v are powers of the same word.*

Recall that a word is said to be *primitive* if it is not a non-trivial power of some word. The following very simple lemma will also be of use.

Lemma 6.1.19. *Let $u, v \in \Sigma^+$ be conjugates of one another. Then u is primitive if and only if v is primitive.*

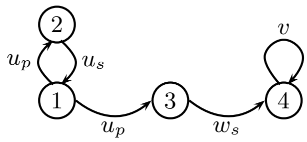
Proof. Let $u, v \in \Sigma^+$ be conjugates of one another, say $u = xy$ and $v = yx$ with $x, y \in \Sigma^*$. Assume u is not primitive. Then $u = z^r$ for some $z \in \Sigma^+$ and $r \geqslant 2$, and so $u = xy = z^s z_1 z_2 z^t$, where $z_1, z_2 \in \Sigma^*$ and $s, t \geqslant 0$ are such that $z = z_1 z_2$, $x = z^s z_1$, $y = z_2 z^t$, and $s + t + 1 = r$. Then $v = yx = z_2 z^t z^s z_1 = (z_2 z_1)^t (z_2 z_1)^s z_2 z_1 = (z_2 z_1)^{t+s+1}$ is also not primitive. ☐

Lemma 6.1.20. *Let $u, v \in \Sigma^+$ and $w \in \Sigma^*$. Then $F(u^* w v^*) = F({u_c}^* \, w_f \, {v_c}^*)$, for some $u_c, v_c \in \Sigma^+$ and $w_f \in \Sigma^*$, with u_c a conjugate of u, v_c a conjugate of v, and w_f a factor of w which has no proper prefix in common with u_c and no proper suffix in common with v_c.*

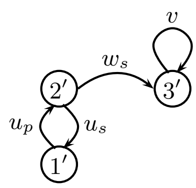
Proof. Let $u, v \in \Sigma^+$ and $w \in \Sigma^*$.

Clearly, we may assume without loss of generality that u and v are primitive.

Suppose u and w have a common proper prefix. Then $u = u_p u_s$ and $w = u_p w_s$, for some $u_p \in \Sigma^+$ and $u_s, w_s \in \Sigma^*$ with u_p the longest common prefix of u and w. Then the language $F(u^* w v^*)$ is recognized by the non-deterministic automaton



in which all states are both initial and final and where, for simplicity, we have not represented the states that join the letters of u_p, u_s, w_s, and v. Carrying out the procedure to construct a deterministic automaton recognizing the same language as a given non-deterministic automaton (see, for example, [20]), we get that $F(u^* w v^*)$ is also recognized by the deterministic automaton



So $F(u^* w v^*) = F({u_c}^* w_s v^*)$, where $u_c = u_s u_p$ is a conjugate of u and the words u_c and w_s do not have a common proper prefix. Now, if v and w_s have a common proper suffix, a similar operation can be carried out and we get $F(u^* w v^*) = F({u_c}^*\, w_f\, {v_c}^*)$, for some $u_c, v_c \in \Sigma^+$ and $w_f \in \Sigma^*$, with u_c a conjugate of u, v_c a conjugate of v, and w_f a factor of w which has no proper prefix in common with u_c and no proper suffix in common with v_c. (Of course, the procedure above may lead to $w_f = \epsilon$, but this constitutes no problem; cf. Section 1.3.) ☐

Also note that, on the other hand, if u and v are conjugate, say $u = z_1 z_2$ and $v = z_2 z_1$, then $F(u^* w v^*) = F(u^* w (z_2 z_1)^*) = F(u^* w z_2 (z_1 z_2)^*)$, that is, $F(u^* w v^*) = F(u^* \tilde{w} u^*)$, with $\tilde{w} \in \Sigma^*$.

The characterization of the ultimately periodic languages for which the associated inverse semigroup is finitely presented is given in the following theorem.

Theorem 6.1.21. *Let Σ be a finite alphabet. Let $L = F(u^* w v^*)$, with $u, v \in \Sigma^+$ primitive words and $w \in \Sigma^*$, be a non-periodic language. Then the tiling semigroup $S(L)$ is finitely presented as an inverse semigroup if and only if u and v are not conjugates of one another.*

Proof. By Theorem 6.1.10, $S(L)$ is finitely presented if and only if the set $M(L)$ of minimal forbidden words of L is finite. We show that $M(L)$ is infinite if and only if u and v are conjugates of one another; in view of Theorem 6.1.10, this yields the desired conclusion.

Assume $M(L)$ is infinite. Then, since Σ is finite, $M(L)$ must contain arbitrarily long words. Then there exist $a, b \in \Sigma$ and $x \in \Sigma^+$ such that ax, $xb \in L$ but $axb \notin L$. Let x be such a word with length greater or equal to $2(|u| + |w| + |v|) - 2$.

Since the words belonging to L are those which can be read as factors of $\dots uuuuwvvv \dots$, a long enough word belonging to L must contain "many" copies of u or "many" copies of v (or of both). Thus, one of the following holds: either

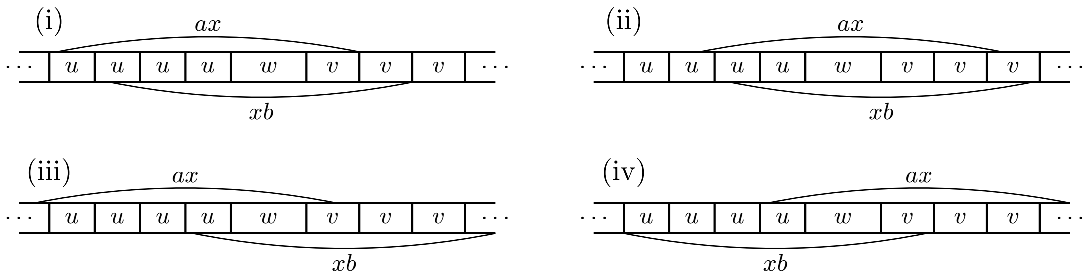
(i) both ax and xb have a conjugate of u as a prefix or

(ii) both ax and xb have a conjugate of v as a suffix

or not, in which case either

(iii) ax uses at most the first v and xb uses at most the last u or

(iv) ax uses at most a the last u and xb uses at most the first v.



Thus,

(i)

$$ax = u_1 z_1 \text{ where } u_1 \in F(u^*),\ |u_1| = |u|, \text{ and } z_1 \in \Sigma^+$$
$$xb = u_2 z_2 \text{ where } u_2 \in F(u^*),\ |u_2| = |u|, \text{ and } z_2 \in \Sigma^+$$

(ii)

$$ax = z_1 v_1 \text{ where } v_1 \in F(v^*),\ |v_1| = |v|, \text{ and } z_1 \in \Sigma^+$$
$$xb = z_2 v_2 \text{ where } v_2 \in F(v^*),\ |v_2| = |v|, \text{ and } z_2 \in \Sigma^+$$

(iii)

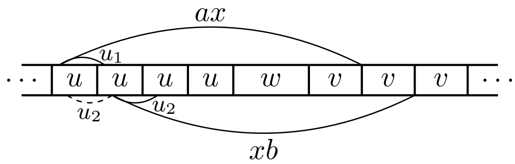
$$ax = u_1 z_1 \text{ where } u_1 \in F(u^*) \text{ and } z_1 \text{ is a proper prefix of } wv$$
$$xb = z_2 v_1 \text{ where } v_1 \in F(v^*) \text{ and } z_2 \text{ is a proper suffix of } uw$$

(iv)

$$ax = z_1 v_1 \text{ where } v_1 \in F(v^*) \text{ and } z_1 \text{ is a proper suffix of } uw$$
$$xb = u_1 z_2 \text{ where } u_1 \in F(u^*) \text{ and } z_1 \text{ is a proper prefix of } wv\,.$$

Cases (ii) and (iv) are analogous to (i) and (iii), so we deal only with these two.

Suppose (i). First note that $u_1, u_2 \in F(u^*)$ and $|u| = |u_1| = |u_2|$ imply that u_1 and u_2 are in fact conjugates of u, from Lemma 6.1.12.



We claim that the last letter of u_2 is a; this being the case, axb can be read in the bi-infinite string as an extension of xb to the left, so that $axb \in L$, a contradiction.

Let $u_1 = au'$. Then $x = u'z_1$ so that $u_2 z_2 = xb = u'z_1 b$. Thus, $u_2 = u'c$, for some $c \in \Sigma$ as $|u'| = |u_1| - 1 = |u_2| - 1$. Since u_1 and u_2 are conjugates of u, we have $u_1\chi = u_2\chi$ where χ is the homomorphism of Σ^* onto the free commutative monoid on Σ such that $s\chi = s$ for each $s \in \Sigma$ (see the proof of Proposition 6.1.13). Thus

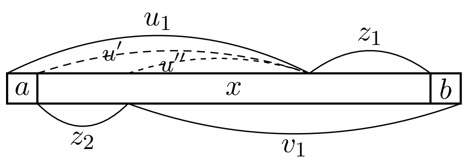
$$a\chi u'\chi = u_1\chi = u_2\chi = u'\chi c\chi$$

so that, since the free commutative monoid is cancellative, $a\chi = c\chi$. Therefore, $a = c$, and so a is the last letter of u_2. It follows that $axb = au_2z_2 \in L$. Hence, case (i) cannot occur when $axb \notin L$.

Now suppose (iii). We claim that u and v are conjugate. Again, $ax = u_1 z_1$ implies that $u_1 = au'$ for some $u' \in \Sigma^*$. As z_1 is a proper factor of wv, we have $|z_1| \leqslant |w| + |v| - 1$. Similarly, $|z_2| \leqslant |u| + |w| - 1$. Since $|x| \geqslant 2(|u| + |w| + |v|) - 2$, then

$$\begin{aligned}
|u'| - |z_2| &= (|x| - |z_1|) - |z_2| \\
&\geqslant 2(|u| + |w| + |v|) - 2 - |z_1| - |z_2| \\
&\geqslant 2(|u| + |w| + |v|) - 2 - (|v| + |w| - 1) - (|u| + |w| - 1) \\
&= |u| + |v| \\
&> |u| + |v| - \gcd(|u|, |v|)\,.
\end{aligned}$$

Thus, u' is longer than z_2. It follows that $u' = z_2 u''$, for some $u'' \in \Sigma^*$ with length greater than $|u| + |v| - \gcd(|u|, |v|)$.

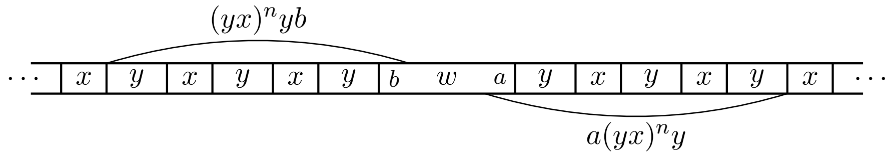


From $u_1 = au' = az_2u''$, we get that $u'' \in F(u^*)$, and so $u'' = {u_c}^r u_p$, for some u_c conjugate of u, u_p prefix of u, and positive integer r, as $|u''| > |u| + |v| - \gcd(|u|, |v|) \geqslant |u|$. Likewise, from $v_1 = u''z_1 b$, we get that $u'' \in F(v^*)$, and so $u'' = {v_c}^t v_p$, for some v_c conjugate of v, v_p prefix of v, and positive integer t. Thus, u'' is a prefix of both ${u_c}^{r+1}$ and ${v_c}^{t+1}$ of length greater than

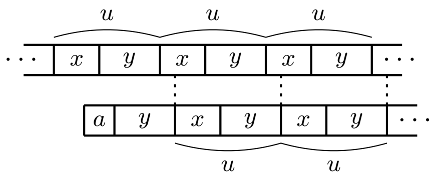
$|u| + |v| - \gcd(|u|, |v|)$, and so, by Proposition 6.1.18, it follows that u_c and v_c are powers of a common word. But, by Lemma 6.1.19, both u_c and v_c are primitive as u and v are primitive words, so the only possibility is $u_c = v_c$. Therefore, u and v are conjugate.

Conversely, suppose that u and v are conjugates of one another, say $u = xy$ and $v = yx$, with $x \in \Sigma^+$ and $y \in \Sigma^*$. By Lemma 6.1.20, we may assume that u and w do not have a common prefix and that v and w do not have a common suffix. (Note that this does not change the fact that u and v are conjugates, since conjugacy is a transitive relation.)

Suppose w is not the empty word. Let a be the last letter of w. Then a is not the last letter of v, and so a is not the last letter of x. Similarly, let b be the first letter of w. Then b is not the first letter of u, and so, of x. For each positive integer n, consider the word $a(yx)^n yb$. We show that, for all but possibly a finite number of (small) positive integers, $a(yx)^n yb$ is a minimal forbidden word for L.

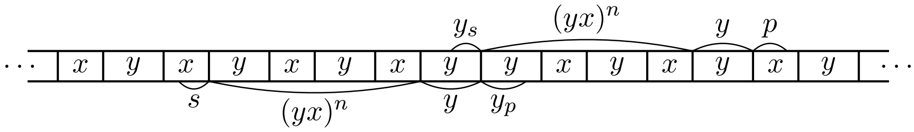


It is obvious that $a(yx)^n y, (yx)^n yb \in L$, for all $n \geqslant 1$, because $a(yx)^n y = av^n y$ is a factor of wv^{n+1} and $(yx)^n yb = y(xy)^n b = yu^n b$ is a factor of $u^{n+1}w$. In order to obtain a contradiction, suppose $a(yx)^n yb \in L$. Note that $a(yx)^n yb$ cannot be read starting at the last letter of w (as an extension of $a(yx)^n y$ to the right) nor ending at its first letter (as an extension of $(yx)^n yb$ to the left), as b is not the first letter of x and a is not the last letter of x, respectively. We show that it cannot be read anywhere else in the bi-infinite word either. Since b is not the first letter of x, the word $a(yx)^n yb$ would have to occur in the bi-infinite word in such a way that its last letter is not the first letter of a copy of x; likewise, since a is not the last letter of x, the first letter of $a(yx)^n yb$ cannot be the last letter of a copy of x. Also, for n large enough $a(yx)^n yb$ contains many powers of xy to the left of w or many powers of yx to its right. Suppose that there are many powers of xy to the left of w. As we have just noted, it cannot be that these occurrences of xy match with occurrences of u, as it would imply that ay was a suffix of u and so a the last letter of x.



So the occurrences of xy will be interior factors of uu, that is, u is an interior factor of uu. Then $uu = u_p u u_s$, where $u_p \in \Sigma^+$ is a prefix of u and $u_s \in \Sigma^+$ is a suffix of u. But since $|u_p| + |u_s| = |u|$, we have $u = u_p u_s$. So $u_p u u_s = uu = u_p u_s u_p u_s$ and thus $u_p u_s = u = u_s u_p$. By Proposition 6.1.17, u_p and u_s are powers of the same word and therefore so is u, a contradiction since by assumption u is primitive. Similarly, if there are many powers of yx to the right of w, then v is not primitive, again a contradiction. We conclude that $a(yx)^n yb \notin L$, for infinitely many positive integers n.

Now suppose w is the empty word. Then $L = F(u^*v^*) = F((xy)^*(yx)^*)$ with both x and y non-empty and $xy \neq yx$, for otherwise the bi-infinite word would be periodic. Thus, y is not a prefix nor a suffix of x, nor vice-versa. Let y_p be the shortest prefix of y which is not a prefix of x and y_s the shortest suffix of y which is not a suffix of x. Since y_p and y_s are at least one-letter long, write $y_p = pa$ and $y_s = bs$, where $a, b \in \Sigma$ and $p, s \in \Sigma^*$. Then p is a common prefix of x and y and s is a common suffix of x and y. Consider the word $y_s(yx)^n yy_p$ with n a positive integer.



Its longest proper factors are $y_s(yx)^n yp$ and $s(yx)^n yy_p$ and both occur in the bi-infinite word, since $y_s(yx)^n yp$ is a factor of $y(yx)^{n+1}$ and $s(yx)^n yy_p$ is a factor of $x(yx)^n yy$, that is, of $(xy)^{n+1}y$. However, $y_s(yx)^n yy_p$ does not occur in the bi-infinite word. As argued in the case $w \neq \epsilon$, $y_s(yx)^n yy_p$ cannot occur as a factor of $y(yx)^{n+1}$ or as a factor of $(xy)^{n+1}y$ with the occurrences of xy aligned with the occurrences of xy in the bi-infinite word, since y_p is not a prefix and y_s is not a suffix of x, also it cannot occur with these occurrences not aligned, since it implies that u and v are not primitive. Therefore, $y_s(yx)^n yy_p$ is a minimal forbidden word for L, for all positive integers.

Thus, $M(L)$ is infinite in either case and hence $S(L)$ is not finitely presented by Theorem 6.1.10. ☐

In view of Lemma 6.1.20, as a consequence of Theorem 6.1.21 we have that the inverse semigroup associated with an ultimately periodic language is finitely presented if and only if the underlying bi-infinite word cannot be written in the form $\cdots uuu\tilde{w}uuu \cdots$. Also,

Corollary 6.1.22. *Let Σ be a finite alphabet. Let $L = F(u^*wv^*)$, with $u, v \in \Sigma^+$ primitive words and $w \in \Sigma^*$. If the tiling semigroup $S(L)$ is finitely presented, then the minimal forbidden words of L have length less than $2(|u| + |v| + |w|)$.*

Proof. This is a consequence of the proof of Theorem 6.1.21. In fact, if L has minimal forbidden words of length greater than $2(|u|+|v|+|w|)$, it follows from the proof of Theorem 6.1.21 that u and v are conjugates, and, therefore, $S(L)$ is infinitely presented. ☐

Unfortunately, our study failed to cover all factorial languages — in fact, it fails to cover all regular tiling languages — since, as noticed in Chapter 1, Section 1.3, periodic and ultimately periodic languages do not exhaust the class of (regular) languages consisting of factors of bi-infinite words. For now, the most that can be said is that the inverse semigroup associated with a factorial language which is neither periodic not ultimately periodic can be finitely (cf. Example 6.1.2) or infinitely presented (cf. Example 6.1.11).

6.2 Infinite presentability of n-dimensional hypercubic tiling semigroups with $\mathbf{n} \geqslant 2$

It is well-known that the tiling semigroup $S(\mathcal{T})$ associated with an arbitrary n-dimensional tiling $\mathcal{T}$ is a strongly E^*-unitary inverse semigroup with zero (see, for example, [58]), by means of the 0-restricted idempotent-pure pre-homomorphism λ of $S(\mathcal{T})$ into the abelian group with zero $(\mathbb{R}^n, +)^{\overline{0}}$ that maps 0 to $\overline{0}$ and each non-zero element $[a, A, b]$ to the vector v_{ab} with initial point in the centre of the tile a and terminal point in the centre of b. (In the case of an n-dimensional hypercubic tiling, or, more generally, in the case of the inverse semigroup associated with an n-dimensional factorial language, we can obviously take $\lambda\colon S(\mathcal{T}) \to \mathbb{Z}^n$; in fact, even in the case of an arbitrary n-dimensional (finite type) tiling, it is possible to take $\lambda\colon S(\mathcal{T}) \to \mathbb{Z}^m$ for some positive integer m, since the fact that $S(\mathcal{T})$ is finitely generated and $(\mathbb{R}^n, +)$ a torsion-free, abelian group imply that $(S(\mathcal{T}))\,\lambda$ is a finitely generated torsion-free abelian group (see, for example, [30, Theorem 8.4]).)

As noted in Section 1.2 (cf. Remark 1.2.1), this has the following consequence: for all x and y in a tiling semigroup $S(\mathcal{T})$, the product $xyx^{-1}y^{-1}$ is an idempotent in $S(\mathcal{T})$ and, thus, if $Inv_0\,\langle A \mid R\rangle$ is a presentation for a tiling semigroup, then all relations of the form $(uvu^{-1}v^{-1})^2 = uvu^{-1}v^{-1}$, with $u, v \in (A \cup A^{-1})^+$, have to be a consequence of the relations in R. Notice that this does not mean that a tiling semigroup has to be infinitely presented as an inverse semigroup; for example for one-dimensional tiling semigroups, we know from the previous section that this depends on the minimal forbidden words of the language associated with the tiling.

In this section, we prove that each n-dimensional hypercubic tiling with $n \geqslant 2$ is infinitely presented as an inverse semigroup. In fact, we will show that an n-dimensional hypercubic tiling $\mathcal{T}$ with $n \geqslant 2$ is always infinitely presented as a strongly E^*-unitary inverse semigroup which admits a 0-restricted idempotent-pure pre-homomorphism into an abelian group with zero. Recall from Chapter 1 (cf. Definition 1.2.3) that this means that $S(\mathcal{T})$ requires an infinite number of relations beyond those that ensure that all elements of the form $xyx^{-1}y^{-1}$ are idempotents. We will start with the two-dimensional case, and then generalize our findings to arbitrary dimension.

In [44], a detailed proof for the two-dimensional single-coloured hypercubic tiling semigroup is given and the generalization to the n-dimensional case outlined, but the colour generalization is only mentioned. There, the strategy was to represent the tiling semigroup as a quotient of a well-known construction, the graph expansion of a group presentation (see [36]); here, we will develop a close analogue of the Margolis–Meakin construction to deal with the tiling semigroup of an arbitrary (that is, possibly coloured) two-dimensional hypercubic tiling. Our strategy is, given a two-dimensional hypercubic tiling semigroup $S(\mathcal{T})$, to construct a finitely generated strongly E^*-unitary inverse semigroup $W_{\mathcal{T}}$ with zero that admits a 0-restricted idempotent-pure pre-homomorphism into an abelian group with zero and to show that $S(\mathcal{T}) \simeq W_{\mathcal{T}}/\rho$ for an infinitely generated congruence ρ. In view of Proposition 1.2.8, this yields the desired conclusion.

Recall from Lemma 3.4.6 in Section 3.4 that a two-dimensional hypercubic tiling semigroup $S(\mathcal{T})$ over a finite alphabet Σ is generated, as an inverse semigroup with zero, by the $|\Sigma|$ single-tile idempotents e_i, with $i \in \Sigma$, and the two-tile elements $a_{(i,j)}$ and $b_{(i,j)}$, with $i, j \in \Sigma$, where the vector from the in-tile to the out-tile of $a_{(i,j)}$ is the standard basis vector $(1, 0)$, the vector from the in-tile to the out-tile of $b_{(i,j)}$ is the standard basis vector $(0, 1)$, and such that the underlying finite, connected Σ-coloured subset is a Σ-coloured subset of $\mathcal{T} = (\mathbb{Z}^2, \tau)$ (cf. Figure 3.9).

In the sequel, it will be useful to recall that $e_i\, a_{(i,j)} = a_{(i,j)}$ and $a_{(i,j)}\, e_j = a_{(i,j)}$, for all $i, j \in \Sigma$, and that $e_k\, a_{(i,j)} = 0$ and $a_{(i,j)}\, e_m = 0$, for all $i, j, k, m \in \Sigma$ with $k \neq i$ and $m \neq j$. Similarly, $e_i\, b_{(i,j)} = b_{(i,j)}$ and $b_{(i,j)}\, e_j = b_{(i,j)}$, for all $i, j \in \Sigma$, and $e_k\, b_{(i,j)} = 0$ and $b_{(i,j)}\, e_m = 0$, for all $i, j, k, m \in \Sigma$ with $k \neq i$ and $m \neq j$.

For motivation, we begin by recalling some notions from the Margolis–Meakin construction. Let G be a group, $Gp\,\langle X|R\rangle$ a group presentation for G, and $\Gamma(X;R)$ the Cayley graph of the presentation $Gp\,\langle X|R\rangle$ (cf. Section 3.4 for the definition of Cayley graph of a group presentation). For each finite connected subgraph Γ of $\Gamma(X;R)$ and each $g \in G$, let $g \cdot \Gamma$ be the subgraph of $\Gamma(X;R)$ obtained by acting Γ on the left by g, that is, the graph with $V(g\cdot\Gamma) = \{gh \colon h \in V(\Gamma)\}$ and $E(g\cdot\Gamma) = \{(gh, ghx) \colon (h, hx) \in E(\Gamma)\}$. The *Cayley graph expansion of the group presentation $Gp\,\langle X|R\rangle$* is then the monoid $M(X;R)$ with elements (Γ, g), where Γ is a finite, non-empty, connected subgraph of $\Gamma(X;R)$ with $1, g \in V(\Gamma)$, and multiplication given by

$$(\Gamma, g)(\Delta, h) = (\Gamma \cup (g \cdot \Delta), gh)\,.$$

The following summarizes some of the results obtained by Margolis and Meakin in [36], where σ and $\sigma^\natural$ denote, as usual, the minimum group congruence and the canonical epimorphism associated with σ, respectively. Where necessary, a subscript will be added to clarify on which monoid the minimum group congruence is defined.

Theorem 6.2.1 ([36], Theorem 2.1, Theorem 2.2, Corollary 2.9, and Lemma 3.2)**.** *In the notation above, $M(X;R)$ is an E-unitary monoid generated by X as an inverse monoid with maximum group image G. Furthermore,*

(i) (Γ, g) *is idempotent if and only if* $g = 1$*;*

(ii) $(\Gamma, g)\sigma = (\Delta, h)\sigma$ *if and only if* $g = h$*;*

(iii) $M(X;R)$ *satisfies the following universal property: given any E-unitary monoid M generated by X as an inverse monoid, with maximum group image G, there exists a unique idempotent-pure epimorphism $\varphi\colon M(X;R) \twoheadrightarrow M$ such that $\varphi\sigma^\natural_M = \sigma^\natural_{M(X;R)}$;*

(iv) $M(X;R)$ *is defined by the inverse monoid presentation*

$$Inv\left\langle X \,|\, u^2 = u \ (u \in FIM_X \text{ and } u = 1 \text{ in } G)\right\rangle\,.$$

In particular, consider the usual group presentation for $\mathbb{Z}^2$, namely $\mathbb{Z}^2 = Gp\,\langle X|C\rangle$, with $X = \{a, b\}$, say, and $C = \{x + y - x - y = 0 \colon x, y \in X\}$. Notice that, for each finite connected subgraph Γ of $\Gamma(X; C)$ and each $g \in \mathbb{Z}^2$, the action $g \cdot \Gamma$ on Γ on the left by g yields the graph with vertex set $V(g \cdot \Gamma) = \{g + h = h + g \colon h \in V(\Gamma)\}$ and set of edges

$$E(g \cdot \Gamma) = \{(g + h, g + h + x) = (h + g, h + x + g) \colon (h, h + x) \in E(\Gamma)\}\ ,$$

which we will denote more naturally by $\Gamma + g$.

In the case of the two-dimensional single-coloured hypercubic tiling semigroup, the zero can be omitted and the semigroup can be regarded as an E-unitary inverse semigroup, instead of a strongly E^*-unitary inverse semigroup. Also, the tiling semigroup has two non-idempotent generators and maximum group image $\mathbb{Z}^2$. In [44], we explicitly described the kernel of the epimorphism φ in Theorem 6.2.1 (iii) from the Margolis–Meakin expansion $M(X; C)$, with X and C as in the previous paragraph, onto the two-dimensional single-coloured hypercubic tiling semigroup (Proposition 3.2) and proved that it is an infinitely generated congruence (Proposition 3.3), to conclude that the two-dimensional single-coloured hypercubic tiling semigroup is infinitely presented, even as an E-unitary inverse semigroup with abelian maximum group image (Theorem 3.4). We shall now adapt this construction to the case of an arbitrary two-dimensional hypercubic tiling semigroup.

Definition 6.2.2. Let Σ be an alphabet. A pair (Γ, α), where Γ is a subgraph of the Cayley graph $\Gamma(X; C)$ of the usual group presentation of $\mathbb{Z}^2$ and $\alpha \colon V(\Gamma) \to \Sigma$ is a map, is called a Σ-*coloured subgraph of* $\mathbb{Z}^2$. We say that (Γ, α) is *finite*, or *connected*, if Γ is finite, or connected, respectively.

Note that, in considering such pairs (Γ, α), we are simply assigning colours to the vertices of Γ and that, for each (finite, connected) Σ-coloured subgraph (Γ, α) of $\mathbb{Z}^2$, we have that $(V(\Gamma), \alpha)$ is a (finite, connected) Σ-coloured subset of $\mathbb{Z}^2$.

Similarly to the action on the subgraphs by elements of the group considered by Margolis and Meakin, for each Σ-coloured subgraph (Γ, α) of $\mathbb{Z}^2$ and each $z \in \mathbb{Z}^2$, we set that $(\Gamma, \alpha) + z$ is the Σ-coloured subgraph $(\Gamma + z, \beta)$ of $\mathbb{Z}^2$ with $\beta \colon V(\Gamma + z) \to \Sigma$ defined by $\beta(g + z) = \alpha(g)$ for each $g \in V(\Gamma)$. As in Chapter 4, we write $\alpha + g$ to denote this mapping β.

In what follows, unless the contrary is explicitly mentioned, $\mathcal{T} = (\mathbb{Z}^2, \tau)$ will denote an arbitrary, but fixed, two-dimensional hypercubic tiling with colours in a finite set Σ. Relative to $S(\mathcal{T})$, consider the following construction:

Theorem 6.2.3. *Let $W_{\mathcal{T}}$ be the set consisting of $\tilde{0}$ together with all pairs $((\Gamma, \alpha), g)$, where (Γ, α) is a finite, connected Σ-coloured subgraph of $\mathbb{Z}^2$ such that $(V(\Gamma), \alpha) + z$ is a Σ-coloured subset of $(\mathbb{Z}^2, \tau)$ for some $z \in \mathbb{Z}^2$, with $0, g \in V(\Gamma)$. Consider the following operation on $W_{\mathcal{T}}$: for all $((\Gamma, \alpha), g), ((\Delta, \beta), h) \in W_{\mathcal{T}}$,*

$$((\Gamma, \alpha), g)((\Delta, \beta), h) = ((\Gamma, \alpha) \cup ((\Delta, \beta) + g), g + h)\,,$$

if $\big((V(\Gamma),\alpha)\cup((V(\Delta),\beta)+g)\big)+z$ *is a finite, connected* Σ*-coloured subset of* $(\mathbb{Z}^2,\tau)$ *for some* $z\in\mathbb{Z}^2$*, and* $\tilde{0}$ *otherwise; all products involving* $\tilde{0}$ *yield* $\tilde{0}$*. Then* $W_{\mathcal{T}}$ *is a strongly* E^**-unitary inverse semigroup which admits a* 0*-restricted idempotent-pure pre-homomorphism into an abelian group with zero.*

Proof. Let $((\Gamma,\alpha),g),((\Delta,\beta),h)\in W_{\mathcal{T}}$ be such that $((\Gamma,\alpha),g)((\Delta,\beta),h)\neq\tilde{0}$. Then, by definition, $\big((V(\Gamma),\alpha)\cup((V(\Delta),\beta)+g)\big)+z$ is a finite, connected Σ-coloured subset of $(\mathbb{Z}^2,\tau)$, for some $z\in\mathbb{Z}^2$. Since

$$\big(((V(\Gamma),\alpha)\cup((V(\Delta),\beta)+g))\big)+z=(V(\Gamma)\cup(V(\Delta)+g),\alpha\cup(\beta+g))+z\,,$$

it follows that $\alpha|_{V(\Gamma)\cap(V(\Delta)+g)}=(\beta+g)|_{V(\Gamma)\cap(V(\Delta)+g)}$, and so $((\Gamma,\alpha)\cup((\Delta,\beta)+g),g+h)$ is a well-defined finite Σ-coloured subgraph of $\mathbb{Z}^2$. Also, $((\Gamma,\alpha)\cup((\Delta,\beta)+g),g+h)$ is connected since both $((\Gamma,\alpha),g)$ and $((\Delta,\beta),h)$ are connected and $g=0+g$ belongs to $V(\Gamma)\cap(V(\Delta)+g)$. Moreover, $0\in V(\Gamma)$ implies that $0\in V(\Gamma)\cup(V(\Delta)+g)$ and $h\in V(\Delta)$ implies that $g+h=h+g\in V(\Gamma)\cup(V(\Delta)+g)$. Thus, $((\Gamma,\alpha),g)((\Delta,\beta),h)\in W_{\mathcal{T}}$, and so the operation is well-defined.

As noted after Definition 6.2.2, $(V(\Gamma),g)$ is a finite, connected Σ-coloured subset of $\mathbb{Z}^n$ for each finite, connected Σ-coloured (Γ,g) subgraph of $\mathbb{Z}^n$. So, $((\Gamma,\alpha),g),((\Delta,\beta),h)\in W_{\mathcal{T}}$ implies that $(V(\Gamma),\alpha)$ and $(V(\Delta),\beta)$ — and, thus, $(V(\Delta),\beta)+g=(V(\Delta)+g,\beta+g)$ — are finite, connected Σ-coloured subsets of $\mathbb{Z}^2$. It follows that $(V(\Gamma),\alpha)\cup(V(\Delta)+g,\beta+g)$ is also a finite, connected Σ-coloured subset of $\mathbb{Z}^2$ provided that $\alpha|_{V(\Gamma)\cap(V(\Delta)+g)}=(\beta+g)|_{V(\Gamma)\cap(V(\Delta)+g)}$, since $V(\Gamma)\cup(V(\Delta)+g)$ is obviously finite and it is connected as $g\in V(\Gamma)\cap(V(\Delta)+g)$. Therefore,

$$\big((V(\Gamma),\alpha)\cup(V(\Delta)+g,\beta+g)\big)+z=((V(\Gamma)+z)\cup(V(\Delta)+g+z),(\alpha+z)\cup(\beta+g+z))$$

is a finite, connected Σ-coloured subset of $(\mathbb{Z}^2,\tau)$ provided that $(\alpha+z)\cup(\beta+g+z)$ is the restriction of τ to $(V(\Gamma)+z)\cup(V(\Delta)+g+z)$. We conclude that

$$((\Gamma,\alpha),g)((\Delta,\beta),h)\neq\tilde{0}\Leftrightarrow\exists\,z\in\mathbb{Z}^2,\ \tau|_{(V(\Gamma)+z)\cup(V(\Delta)+g+z)}=(\alpha+z)\cup(\beta+g+z)\,. \tag{6.4}$$

To prove associativity, let $((\Gamma,\alpha),g),((\Delta,\beta),h),((\Pi,\gamma),k)\in W_{\mathcal{T}}$.

We first check that the product $\big(((\Gamma,\alpha),g)((\Delta,\beta),h)\big)((\Pi,\gamma),k)$ is non-zero if and only if the product $((\Gamma,\alpha),g)\big(((\Delta,\beta),h)((\Pi,\gamma),k)\big)$ is non-zero. By (6.4), we have:

$$\begin{aligned}
&\big(((\Gamma,\alpha),g)((\Delta,\beta),h)\big)((\Pi,\gamma),k)\neq\tilde{0}\Leftrightarrow\\
&\Leftrightarrow\exists\,z\in\mathbb{Z}^2,\ \tau|_{(V(\Gamma)+z)\cup(V(\Delta)+g+z)}=(\alpha+z)\cup(\beta+g+z)\text{ and}\\
&\qquad\qquad\qquad\qquad((\Gamma,\alpha)\cup((\Delta,\beta)+g),g+h)((\Pi,\gamma),k)\neq\tilde{0}\\
&\Leftrightarrow\exists\,w\in\mathbb{Z}^2,\ \tau|_{((V(\Gamma)\cup(V(\Delta)+g))+w)\cup(V(\Pi)+g+h+w)}=\big((\alpha\cup(\beta+g))+w\big)\cup(\gamma+g+h+w)\,,
\end{aligned}$$

since this condition already implies that $\tau|_{(V(\Gamma)+z)\cup(V(\Delta)+g+z)} = (\alpha+z)\cup(\beta+g+z)$ with $z=w$. Thus,

$$\begin{aligned}
&\big(((\Gamma,\alpha),g)((\Delta,\beta),h)\big)((\Pi,\gamma),k) \neq \tilde{0} \Leftrightarrow \\
&\Leftrightarrow \exists\, w\in\mathbb{Z}^2,\ \tau|_{(V(\Gamma)+w)\cup(V(\Delta)+g+w)\cup(V(\Pi)+g+h+w)} = (\alpha+w)\cup(\beta+g+w)\cup(\gamma+g+h+w)\\
&\Leftrightarrow \exists\, w\in\mathbb{Z}^2,\ \tau|_{(V(\Gamma)+w)\cup((V(\Delta)\cup(V(\Pi)+h))+g+w)} = (\alpha+w)\cup\big((\beta\cup(\gamma+h))+g+w\big)\,,
\end{aligned}$$

which implies, in particular, that $\tau|_{(V(\Delta)+g+w)\cup(V(\Pi)+g+h+w)} = (\beta+g+w)\cup(\gamma+g+h+w)$, that is, $((\Delta,\beta),h)((\Pi,\gamma),k)\neq\tilde{0}$, so that

$$\big(((\Gamma,\alpha),g)((\Delta,\beta),h)\big)((\Pi,\gamma),k)\neq\tilde{0} \Leftrightarrow ((\Gamma,\alpha),g)\big(((\Delta,\beta),h)((\Pi,\gamma),k)\big)\neq\tilde{0}\,,$$

as claimed.

In the case of a non-zero product, we have

$$\begin{aligned}
\big(((\Gamma,\alpha),g)((\Delta,\beta),h)\big)((\Pi,\gamma),k) &= \big((\Gamma,\alpha)\cup((\Delta,\beta)+g),g+h\big)((\Pi,\gamma),k)\\
&= ((\Gamma,\alpha)\cup((\Delta,\beta)+g)\cup((\Pi,\gamma)+g+h),g+h+k)\\
&= ((\Gamma,\alpha)\cup((\Delta,\beta)\cup((\Pi,\gamma)+h))+g,g+h+k)\\
&= ((\Gamma,\alpha),g)\big(((\Delta,\beta),h)\cup((\Pi,\gamma)+h),h+k\big)\\
&= ((\Gamma,\alpha),g)\big(((\Delta,\beta),h)((\Pi,\gamma),k)\big)\,.
\end{aligned}$$

Thus, the operation is associative.

It is easy to show that a non-zero element $((\Gamma,\alpha),g)$ is an idempotent if and only if $g=0$, and then straightforward to check that the idempotents commute.

To show that $W_{\mathcal{T}}$ is regular, let $((\Gamma,\alpha),g)\in W_{\mathcal{T}}$. Then $((\Delta,\beta),h) = ((\Gamma-g,\alpha-g),-g)$ is a non-zero element of $W_{\mathcal{T}}$: since (Γ,α) is a finite, connected Σ-coloured subgraph of $\mathbb{Z}^2$, so is $(\Gamma-g,\alpha-g)$; since $0\in V(\Gamma)$, then $-g\in V(\Gamma)-g$; as $g\in V(\Gamma)$, then $0\in V(\Gamma)-g$; also $(V(\Gamma),\alpha)+z\leqslant(\mathbb{Z}^2,\tau)$, with $z\in\mathbb{Z}^2$, implies that $(V(\Gamma)-g,\alpha-g)+(g+z)\leqslant(\mathbb{Z}^2,\tau)$, with $g+z\in\mathbb{Z}^2$. Further,

$$\begin{aligned}
((\Gamma,\alpha),g)((\Delta,\beta),h)((\Gamma,\alpha),g) &= ((\Gamma,\alpha),g)((\Gamma-g,\alpha-g),-g)((\Gamma,\alpha),g)\\
&= ((\Gamma,\alpha)\cup((\Gamma-g,\alpha-g)+g)\cup((\Gamma,\alpha)+g-g),g-g+g)\\
&= ((\Gamma,\alpha)\cup(\Gamma,\alpha)\cup(\Gamma,\alpha),g)\\
&= ((\Gamma,\alpha),g)\,.
\end{aligned}$$

Thus, $W_{\mathcal{T}}$ is a regular semigroup.

Therefore, $W_{\mathcal{T}}$ is an inverse semigroup with zero.

Finally, to prove that $W_{\mathcal{T}}$ admits a 0-restricted idempotent-pure pre-homomorphism into an abelian group with zero, simply take the mapping from $W_{\mathcal{T}}$ into $(\mathbb{Z}^2)^{\bar{0}}$ that maps $\tilde{0}$ to $\bar{0}$ and each non-zero element $((\Gamma,\alpha),g)$ to g. It is trivial to check that this mapping the conditions required. ∎

In addition,

Lemma 6.2.4. *The semigroup $W_{\mathcal{T}}$ is finitely generated.*

Proof. Denote by $a = (1, 0)$ and $b = (0, 1)$ the standard basis vectors of $\mathbb{Z}^2$. Consider the following non-zero elements of $W_{\mathcal{T}}$:

- $E_i = ((\Gamma_0, \alpha_i), 0)$, where $V(\Gamma_0) = \{0\}$ and $\alpha_i(0) = i$;

- $G_{a;\,i,j} = ((\Gamma_a, \alpha_{a;\,i,j}), a)$, where $V(\Gamma_a) = \{0, a\}$, $\alpha_{a;\,i,j}(0) = i$ and $\alpha_{a;\,i,j}(a) = j$;

- $G_{b;\,i,j} = ((\Gamma_b, \alpha_{b;\,i,j}), b)$, where $V(\Gamma_b) = \{0, b\}$, $\alpha_{b;\,i,j}(0) = i$ and $\alpha_{b;\,i,j}(b) = j$,

where $i, j \in \Sigma$. Note that $V(\Gamma_0) = \{0\}$ implies that $E(\Gamma_0) = \emptyset$, $V(\Gamma_a) = \{0, a\}$ implies that $E(\Gamma_a) = \{(0, a)\}$, and $V(\Gamma_b) = \{0, b\}$ implies that $E(\Gamma_b) = \{(0, b)\}$.

We claim that the set $Y = \{E_i \in W_{\mathcal{T}} \colon i \in \Sigma\} \cup \{G_{a;i,j}, G_{b;i,j} \in W_{\mathcal{T}} \colon i, j \in \Sigma\}$ generates $W_{\mathcal{T}}$ as an inverse semigroup with zero. Let $((\Gamma, \alpha), g)$ be a non-zero element of $W_{\mathcal{T}}$. If $((\Gamma, \alpha), g) = E_i$ for some $i \in \Sigma$, then $((\Gamma, \alpha), g)$ is trivially a product of elements from Y, so we may assume otherwise.

Claim 1. *If Γ is a connected path and 0 and g its endpoints, then $((\Gamma, \alpha), g)$ is a product of elements from $Y \cup Y^{-1}$.*

Suppose Γ is the path with $E(\Gamma) = \{(h_0, h_1), (h_1, h_2), \dots, (h_{m-1}, h_m)\}$ with $h_0 = 0$ and $h_m = g$. Since Γ is a subgraph of $\mathbb{Z}^2$, then $h_k - h_{k-1} = \pm a$ or $h_k - h_{k-1} = \pm b$, for all $k \in [m]$. Thus, each pair $((\{0, h_k - h_{k-1}\}, \beta_k), h_k - h_{k-1})$, where $\beta_k(0) = \alpha(h_{k-1})$ and $\beta_k(h_k - h_{k-1}) = \alpha(h_k)$, is either an element of the form $G_{a;\,i,j}$ or $G_{b;\,i,j}$ or an inverse of one of these elements. (Note that, for simplicity, we are identifying the graph with vertex set $\{0, h_k - h_{k-1}\}$ and only edge $(0, h_k - h_{k-1})$ with its vertex set, and that there is no ambiguity in doing so.) Moreover, direct calculations show that

$$
\begin{aligned}
&\prod_{k=1}^{m} ((\{0, h_k - h_{k-1}\}, \beta_k), h_k - h_{k-1}) = \\
&= ((\{0, h_1\}, \beta_1), h_1)((\{0, h_2 - h_1\}, \beta_2), h_2 - h_1) \left(\prod_{k=3}^{m} ((\{0, h_k - h_{k-1}\}, \beta_k), h_k - h_{k-1}) \right) \\
&= ((\{0, h_1\} \cup (\{0, h_2 - h_1\} + h_1), \beta_1 \cup (\beta_2 + h_1)), h_1 + h_2 - h_1) \\
&\qquad \left(\prod_{k=3}^{m} ((\{0, h_k - h_{k-1}\}, \beta_k), h_k - h_{k-1}) \right) \\
&= ((\{0, h_1, h_2\}, \alpha|_{\{0, h_1, h_2\}}), h_2) \left(\prod_{k=3}^{m} ((\{0, h_k - h_{k-1}\}, \beta_k), h_k - h_{k-1}) \right) \\
&= \dots
\end{aligned}
$$

$$\begin{aligned}
&= ((\{0,h_1,\dots,h_{m-1}\},\alpha|_{\{0,h_1,\dots,h_{m-1}\}}),h_{m-1})((\{0,h_m-h_{m-1}\},\beta_m),h_m-h_{m-1})\\
&= ((\{0,h_1,\dots,h_m\},\alpha|_{\{0,h_1,\dots,h_m\}}),h_m)\\
&= ((\Gamma,\alpha),g)\,.
\end{aligned}$$

Therefore, $((\Gamma,\alpha),g)$ can be written as a product of elements from $Y\cup Y^{-1}$.

Claim 2. *If $g=0$, then $((\Gamma,\alpha),g)$ is a product of elements from $Y\cup Y^{-1}$.*

Since Γ is finite, it can be written as a union $\Gamma=\Delta_1\cup\dots\Delta_m$, with each Δ_k a connected path in Γ which has 0 as one of its endpoints. In this way, every vertex and every edge in Γ is visited at least once by some path Δ_k. Also, each $((\Delta_k,\alpha|_{V(\Delta_k)}),h_k)$, where h_k is the endpoint of Δ_k possibly distinct from 0, is a non-zero element in $W_{\mathcal{T}}$ (because the Σ-coloured subset $(V(\Delta_k),\alpha|_{V(\Delta_k)})$ is a Σ-coloured subset of $(V(\Gamma),\alpha)$) and, by the previous claim, $((\Delta_k,\alpha|_{V(\Delta_k)}),h_k)$ can be written as a product of elements from $Y\cup Y^{-1}$. Thus, by the proof of Theorem 6.2.3, the product

$$\begin{aligned}
((\Delta_k,\alpha|_{V(\Delta_k)}),h_k)\,((\Delta_k,\alpha|_{V(\Delta_k)}),h_k)^{-1} &= ((\Delta_k,\alpha|_{V(\Delta_k)}),h_k)\,((\Delta_k,\alpha|_{V(\Delta_k)})-h_k,-h_k)\\
&= ((\Delta_k,\alpha|_{V(\Delta_k)})\cup((\Delta_k,\alpha|_{V(\Delta_k)})-h_k+h_k),h_k-h_k)\\
&= ((\Delta_k,\alpha|_{V(\Delta_k)}),0)
\end{aligned}$$

can be written as a product of elements from $Y\cup Y^{-1}$ as well. Since

$$\prod_{k=1}^{m}((\Delta_k,\alpha|_{V(\Delta_k)}),0)=((\Delta_1\cup\dots\Delta_m,\alpha|_{V(\Delta_1\cup\dots\Delta_m)}),0)=((\Gamma,\alpha),g)\,,$$

we have our claim.

Now consider an arbitrary element $((\Gamma,\alpha),g)$ (distinct from an element of the form E_i). If $g=0$, then $((\Gamma,\alpha),g)$ can be written as a product of elements from $Y\cup Y^{-1}$ by Claim 2. So suppose $g\neq 0$. Since Γ is connected, we can take a connected path Δ in Γ such that 0 and g are its endpoints. Then $((\Gamma,\alpha),g)=((\Gamma,\alpha),0)\,((\Delta,\alpha|_{V(\Delta)}),g)$ (note that $0,g\in V(\Delta)$, so that $((\Delta,\alpha|_{V(\Delta)}),g)$ is a non-zero element of $W_{\mathcal{T}}$), where both $((\Gamma,\alpha),0)$ and $((\Delta,\alpha|_{V(\Delta)}),g)$ can be written as a product of elements from $Y\cup Y^{-1}$, by Claim 2 and Claim 1, respectively. Therefore, $((\Gamma,\alpha),g)$, too, can be written as a product of elements from $Y\cup Y^{-1}$.

Hence, $W_{\mathcal{T}}$ is generated by Y as an inverse semigroup with zero. Since Σ is finite, so is Y and we conclude that $W_{\mathcal{T}}$ is finitely generated. □

Given a non-zero element $((\Gamma,\alpha),g)$ in $W_{\mathcal{T}}$, we have that, by definition, there exists $z\in\mathbb{Z}^2$ such that $(V(\Gamma),\alpha)+z\leqslant(\mathbb{Z}^2,\tau)$. Thus, $[z,(V(\Gamma),\alpha)+z,g+z]\in S(\mathcal{T})$, as $0\in V(\Gamma)$ implies that $z\in V(\Gamma)+z$ and as $g\in V(\Gamma)$ implies that $g+z\in V(\Gamma)+z$. As usual, we will often write $[z,V(\Gamma)+z,g+z]$ instead of $[z,(V(\Gamma),\alpha)+z,g+z]$, since the colour map of $V(\Gamma)+z$ is determined by τ and z, namely being $\tau|_{V(\Gamma)+z}$. Therefore, we can consider the mapping $\phi\colon W_{\mathcal{T}}\to S(\mathcal{T})$ defined on the non-zero elements by

$$((\Gamma,\alpha),g)\phi=[z,V(\Gamma)+z,g+z]\,,$$

where $z \in \mathbb{Z}^2$ is such that $V(\Gamma) + z \leqslant (\mathbb{Z}^2, \tau)$.

Proposition 6.2.5. *The map ϕ is an epimorphism.*

Proof. The proof is straightforward checking, but for completeness we give the details.

We begin by showing that ϕ is well-defined. As we have already seen, if $((\Gamma, \alpha), g) \in W_{\mathcal{T}}$ and $z \in \mathbb{Z}^2$ is such that $V(\Gamma) + z \leqslant (\mathbb{Z}^2, \tau)$, then $((\Gamma, \alpha), g)\phi \in S(L)$. Now let $((\Gamma, \alpha), g) \in W_{\mathcal{T}}$ and suppose that both z and w in $\mathbb{Z}^2$ are such that $V(\Gamma) + z$ and $V(\Gamma) + w$ are Σ-coloured subsets of $(\mathbb{Z}^2, \tau)$. Then, to conclude that $[z, V(\Gamma) + z, g + z] = [w, V(\Gamma) + w, g + w]$, it suffices to take $y = w - z$ (or $y = z - w$), by definition of the elements in a tiling semigroup. Therefore, ϕ is well-defined.

To show that ϕ is a homomorphism, let $((\Gamma, \alpha), g)$ and $((\Delta, \beta), h)$ be non-zero elements in $W_{\mathcal{T}}$ with, say, $((\Gamma, \alpha), g)\phi = [z, V(\Gamma) + z, g + z]$ and $((\Delta, \beta), h)\phi = [w, V(\Delta) + w, h + w]$, with z and w as in the definition of ϕ.

Assume that the product $((\Gamma, \alpha), g)((\Delta, \beta), h)$ is non-zero. Then

$$\begin{aligned}(((\Gamma, \alpha), g)((\Delta, \beta), h))\phi &= ((\Gamma, \alpha) \cup ((\Delta, \beta) + g), g + h)\phi \\ &= [y, \big(V(\Gamma) \cup (V(\Delta) + g)\big) + y, g + h + y] \,,\end{aligned}$$

where $y \in \mathbb{Z}^2$ is such that $\big(V(\Gamma) \cup (V(\Delta) + g)\big) + y \leqslant (\mathbb{Z}^2, \tau)$. Take $u = y - z$ and $v = g + y - w$. Then,

$$\big(V(\Gamma) + z + u\big) \cup \big(V(\Delta) + w + v\big) = \big(V(\Gamma) + y\big) \cup \big(V(\Delta) + g + y\big) \,,$$

which is a Σ-coloured subset of $(\mathbb{Z}^2, \tau)$, and

$$g + z + u = g + z + y - z = g + y = w + v \,.$$

Therefore, the product $[z, V(\Gamma) + z, g + z][w, V(\Delta) + w, h + w]$ is non-zero by definition of $S(\mathcal{T})$, with

$$\begin{aligned}&[z, V(\Gamma) + z, g + z][w, V(\Delta) + w, h + w] = \\ &\quad = [z + u, (V(\Gamma) + z + u) \cup (V(\Delta) + w + v), h + w + v] \\ &\quad = [y, (V(\Gamma) + y) \cup (V(\Delta) + g + y), h + g + y] \\ &\quad = [y, \big(V(\Gamma) \cup (V(\Delta) + g)\big) + y, g + h + y] \,,\end{aligned}$$

that is, $((\Gamma, \alpha), g)\phi\, ((\Delta, \beta), h)\phi = (((\Gamma, \alpha), g)((\Delta, \beta), h))\phi$.

Now assume that $((\Gamma, \alpha), g)((\Delta, \beta), h) = \tilde{0}$, so that $(((\Gamma, \alpha), g)((\Delta, \beta), h))\phi = 0$. Then $(V(\Gamma) \cup (V(\Delta) + g)) + z$ is not a coloured subset of $(\mathbb{Z}^2, \tau)$, for any $z \in \mathbb{Z}^2$. This means that the matching of colours fails to define a coloured subset of $(\mathbb{Z}^2, \tau)$, and so $((\Gamma, \alpha), g)\phi\, ((\Delta, \beta), h)\phi = 0$ as well. Hence, ϕ is a homomorphism.

Finally, let $[a, A, b] \in S(\mathcal{T})$. Then $A = (A, \tau|_A)$ is a finite, connected coloured Σ-subset of $(\mathbb{Z}^2, \tau)$ and, by definition, this means that A is the vertex set of a finite, connected Σ-subgraph of $\mathbb{Z}^2$, say Γ. Thus, $(\Gamma, \tau|_A) - a$ is a finite, connected subgraph of $\mathbb{Z}^2$ whose vertex set contains 0 and $b - a$, since $a, b \in A = V(\Gamma)$. Therefore, $((\Gamma, \tau_A) - a, b - a)$ is a (non-zero) element of $W_{\mathcal{T}}$ which is mapped under ϕ to $[a, A, b]$. We conclude that ϕ is onto.

Hence, ϕ is an epimorphism. $\square$

We now investigate the congruence $\ker \phi$. For all $((\Gamma, \alpha), g), ((\Delta, \beta), h) \in W_{\mathcal{T}} \setminus \{\tilde{0}\}$, we have $((\Gamma, \alpha), g)\phi = ((\Delta, \beta), h)\phi$ if and only if $[z, V(\Gamma) + z, g + z] = [w, V(\Delta) + w, h + w]$, where $z, w \in \mathbb{Z}^2$ are such that $V(\Gamma) + z, V(\Delta) + w \leqslant (\mathbb{Z}^2, \tau)$. Thus, there exists $y \in \mathbb{Z}^2$ such that $w = z + y$, $V(\Delta) + w = V(\Gamma) + z + y$ and $h + w = g + z + y$. But then $h + w = g + w$, that is, $h = g$, and $V(\Delta) + w = V(\Gamma) + w$, that is, $V(\Delta) = V(\Gamma)$. Note that, under our identification, $V(\Delta) = V(\Gamma)$ means that $(V(\Delta), \beta) = (V(\Gamma), \alpha)$, which implies that $\beta = \alpha$. Therefore,

$$\ker \phi = \left\{(\tilde{0}, \tilde{0})\right\} \cup \{(((\Gamma, \alpha), g), ((\Delta, \alpha), g)) \in W_{\mathcal{T}} \times W_{\mathcal{T}} \colon V(\Gamma) = V(\Delta)\} \ .$$

Next, we establish what will be for us the key property of the map ϕ.

Proposition 6.2.6. *The congruence* $\ker \phi$ *is infinitely generated.*

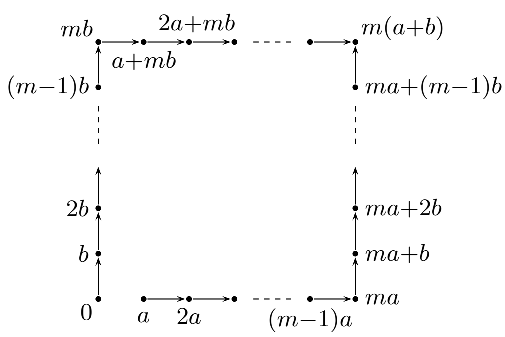
Proof. In order to obtain a contradiction, assume that $\ker \phi$ is generated by the finite set

$$R = \left\{ \big(((\Delta_i, \delta_i), g_i), ((\Omega_i, \delta_i), g_i)\big) \in W_{\mathcal{T}} \times W_{\mathcal{T}} \colon i \in I \right\} \subset \ker \phi.$$

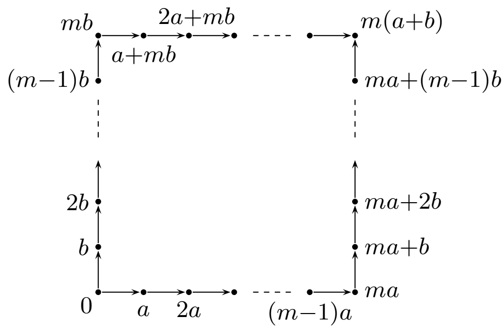
Since each Δ_i (and Ω_i) in a pair belonging to R is a finite graph, we conclude that there cannot exist $\big(((\Delta_i, \delta_i), g_i), ((\Omega_i, \delta_i), g_i)\big) \in R$ such that $V(\Delta_i) = V(\Omega_i)$ has arbitrarily many vertices, as R is a finite set by assumption; take v to be the maximum number of vertices of the graphs of elements in R. For simplicity, assume without loss of generality that $R = R^{-1}$ and that all pairs in R are non-trivial.

Since there exists a unique connected subgraph of $\mathbb{Z}^2$ with three or fewer given vertices, the cases $v = 1$, $v = 2$ and $v = 3$ are trivial, for there are no distinct, $\ker \phi$-related elements $((\Delta_i, \delta_i), g_i)$ and $((\Omega_i, \delta_i), g_i)$ such that $V(\Delta_i) = V(\Omega_i)$ has three or fewer vertices.

Thus, consider $v \geqslant 4$. Let $m = \left\lfloor \frac{v}{2} \right\rfloor$, that is, m is the largest integer smaller or equal to $\frac{v}{2}$. Since $\left\lfloor \frac{v}{2} \right\rfloor = \frac{v}{2}$ if v is even and $\left\lfloor \frac{v}{2} \right\rfloor = \frac{v-1}{2}$ if v is odd, we have $m \geqslant 2$. Let Γ be the finite, non-empty, connected subgraph of $\Gamma(\{a, b\}; C)$



with $4m$ vertices. Note that, since $m \geqslant 2$, the vertex $a+b$ never belongs to $V(\Gamma)$. Let $\alpha\colon V(\Gamma) \to \Sigma$ be any map such that $(V(\Gamma),\alpha)+z$ is a coloured subset of $(\mathbb{Z}^2,\tau)$ for some $z \in \mathbb{Z}^2$. Notice that such mapping always exists; for example, one could take $\alpha = \tau|_{V(\Gamma)}$ (and $z=0$). Consider the element $((\Gamma,\alpha),0)$ of $W_{\mathcal{T}}$. Then $((\Gamma,\alpha),0)\phi = ((\Gamma_c,\alpha),0)\phi$, with Γ_c the graph



Therefore, $V(\Gamma) = V(\Gamma_c)$, but whereas $(0,a)$ is an edge in Γ_c, it is not so in Γ. Then $((\Gamma,\alpha),0)$ and $((\Gamma_c,\alpha),0)$ are distinct $\ker\phi$-related elements of $W_{\mathcal{T}}$, and so there exist a positive integer n and a non-trivial sequence

$$((\Gamma,\alpha),0) = ((\Pi_1,\sigma_1),s_1) \to ((\Pi_2,\sigma_2),s_2) \to \ldots \to ((\Pi_n,\sigma_n),s_n) = ((\Gamma_c,\alpha),0)$$

of elementary R-transitions; that is, there exist elements $((\Pi_1,\sigma_1),s_1),((\Pi_2,\sigma_2),s_2),\ldots$ and $((\Pi_n,\sigma_n),s_n)$ in $W_{\mathcal{T}}$ such that, for each $i \in \{1,\ldots,n-1\}$,

$$\begin{cases} ((\Pi_i,\sigma_i),s_i) = ((\Phi_i,\phi_i),a_i)((\Delta_i,\delta_i),g_i)((\Psi_i,\psi_i),b_i) \\ ((\Pi_{i+1},\sigma_{i+1}),s_{i+1}) = ((\Phi_i,\phi_i),a_i)((\Omega_i,\delta_i),g_i)((\Psi_i,\psi_i),b_i)\,, \end{cases}$$

for some $((\Phi_i,\phi_i),a_i),((\Psi_i,\psi_i),b_i) \in {W_{\mathcal{T}}}^1$ and $\big(((\Delta_i,\delta_i),g_i),((\Omega_i,\delta_i),g_i)\big) \in R$. Notice that, since by assumption all R-transitions are non-trivial and $V(\Delta_i) = V(\Omega_i)$, what happens in each step is the addition, deletion or substitution of at least one edge in the graph $\Delta_i + a_i$ by replacing it by $\Omega_i + a_i$.

Claim 1. *All graphs $\Delta_i + a_i$ are connected subgraphs of Γ_c.*

By definition of the operation in $W_{\mathcal{T}}$, we have

$$\begin{cases} ((\Pi_i,\sigma_i),s_i) = \big((\Phi_i,\phi_i) \cup ((\Delta_i,\delta_i)+a_i) \cup ((\Psi_i,\psi_i)+a_i+g_i), a_i+g_i+b_i\big) \\ ((\Pi_{i+1},\sigma_{i+1}),s_{i+1}) = \big((\Phi_i,\phi_i) \cup ((\Omega_i,\delta_i)+a_i) \cup ((\Psi_i,\psi_i)+a_i+g_i), a_i+g_i+b_i\big)\,. \end{cases}$$

We will show that all graphs Π_i are connected subgraphs of Γ_c (with $V(\Pi_i) = V(\Gamma_c)$); since $\Delta_i + a_i$ is a connected subgraph of Π_i, this yields the desired conclusion.

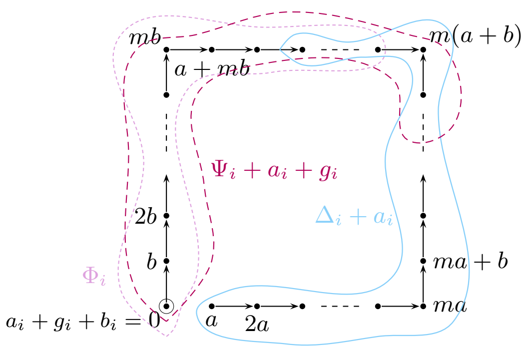
Now,

$$\begin{aligned}
\Gamma = \Pi_1 &= \Phi_1 \cup (\Delta_1 + a_1) \cup (\Psi_1 + a_1 + g_1)\\
\Pi_2 &= \Phi_1 \cup (\Omega_1 + a_1) \cup (\Psi_1 + a_1 + g_1)\\
&= \Phi_2 \cup (\Delta_2 + a_2) \cup (\Psi_2 + a_2 + g_2)\\
\Pi_3 &= \Phi_2 \cup (\Omega_2 + a_2) \cup (\Psi_2 + a_2 + g_2)\\
&= \Phi_3 \cup (\Delta_3 + a_3) \cup (\Psi_3 + a_3 + g_3)\\
&\vdots\\
\Pi_{n-1} &= \Phi_{n-2} \cup (\Omega_{n-2} + a_{n-2}) \cup (\Psi_{n-2} + a_{n-2} + g_{n-2})\\
&= \Phi_{n-1} \cup (\Delta_{n-1} + a_{n-1}) \cup (\Psi_{n-1} + a_{n-1} + g_{n-1})\\
\Gamma_c = \Pi_n &= \Phi_{n-1} \cup (\Omega_{n-1} + a_{n-1}) \cup (\Psi_{n-1} + a_{n-1} + g_{n-1})\,.
\end{aligned}$$

Then Π_1 is a connected subgraph of Γ_c with $V(\Pi_1) = V(\Gamma_c)$, since $\Pi_1 = \Gamma$, Γ is a connected subgraph of Γ_c and $V(\Gamma) = V(\Gamma_c)$. By definition, all Π_i are connected graphs, since $((\Pi_i, \sigma_i), s_i) \in W_{\mathcal{T}}$ for all i. From the fact that $\Gamma = \Pi_1 = \Phi_1 \cup (\Delta_1 + a_1) \cup (\Psi_1 + a_1 + g_1)$, we conclude that Φ_1, $\Delta_1 + a_1$ and $\Psi_1 + a_1 + g_1$ are connected subgraphs of Γ, and so connected subgraphs of Γ_c. Since $\big(((\Delta_1, \delta_1), g_1), ((\Omega_1, \delta_1), g_1)\big) \in R$ implies that $V(\Delta_1) = V(\Omega_1)$, then also $\Omega_1 + a_1$ is a connected subgraph of Γ_c (although not necessarily of Γ), as Γ_c contains all possible edges between its vertices that a subgraph of $\Gamma(\{a, b\}\,; C)$ with vertex set $V(\Gamma_c)$ can contain. Therefore, $\Pi_2 = \Phi_1 \cup (\Omega_1 + a_1) \cup (\Psi_1 + a_1 + g_1)$ is a connected subgraph of Γ_c with $V(\Pi_2) = V(\Gamma_c)$:

$$\begin{aligned}
V(\Pi_2) &= V(\Phi_1) \cup V(\Omega_1 + a_1) \cup V(\Psi_1 + a_1 + g_1)\\
&= V(\Phi_1) \cup V(\Delta_1 + a_1) \cup V(\Psi_1 + a_1 + g_1) = V(\Pi_1) = V(\Gamma_c).
\end{aligned}$$

Now since $\Pi_2 = \Phi_2 \cup (\Delta_2 + a_2) \cup (\Psi_2 + a_2 + g_2)$, we have that Φ_2, $\Delta_2 + a_2$ and $\Psi_2 + a_2 + g_2$ are connected subgraphs of Γ_c and we conclude, as above, that $\Pi_3 = \Phi_2 \cup (\Omega_2 + a_2) \cup (\Psi_2 + a_2 + g_2)$ is a connected subgraph of Γ_c with $V(\Pi_3) = V(\Gamma_c)$. Proceeding in a similar manner for the remaining graphs Π_i, we have our claim.



Notice that $s_i = 0$, for all i. In fact, on the one hand we have that $s_i = a_i + g_i + b_i = s_{i+1}$, for all $i \in [n-1]$; on the other hand, the fact that $((\Gamma,\alpha),0) = ((\Pi_1,\sigma_1),s_1)$ implies that $s_1 = 0$.

To complete Γ to Γ_c, we must add the edge $(0,a)$.

Claim 2. *The edge $(0,a)$ cannot be added using only pairs from R.*

Since $(0,a)$ is an edge in Γ_c but not in Γ, at some point of the sequence

$$((\Gamma,\alpha),0) = ((\Pi_1,\sigma_1),s_1) \to ((\Pi_2,\sigma_2),s_2) \to \ldots \to ((\Pi_n,\sigma_n),s_n) = ((\Gamma_c,\alpha),0)\,,$$

it will be added by an R-transition. Then there is a graph $\Delta_i + a_i$ which contains the vertices 0 and a but not the edge $(0,a)$. Since $\Delta_i + a_i$ is a connected graph, there exists a path in $\Delta_i + a_i$ joining 0 and a; and since $\Delta_i + a_i$ is a connected subgraph of Γ_c by Claim 1, this path is a path in Γ_c. Therefore, there is a path in Γ_c joining the vertices 0 and a and avoiding the edge $(0,a)$. Now, there is a unique such path, and it has all $4m$ vertices of Γ_c. But this cannot be a path in $\Delta_i + a_i$, since Δ_i has at most v vertices and $v < 4\left\lfloor \frac{v}{2} \right\rfloor$. Therefore, it is impossible to add the edge $(0,a)$ to Γ using only pairs $\big(((\Delta_i,\alpha_i),g_i),((\Omega_i,\alpha_i),g_i)\big)$ from R.

Hence, we conclude that no finite set R generates $\ker\phi$. $\square$

Theorem 6.2.7. *Any two-dimensional hypercubic tiling semigroup is infinitely presented, even as a strongly E^*-unitary inverse semigroup that admits a 0-restricted idempotent-pure pre-homomorphism into an abelian group with zero.*

Proof. Let $\mathcal{T} = (\mathbb{Z}^2,\tau)$ be a two-dimensional hypercubic tiling semigroup. Consider the semigroup $W_{\mathcal{T}}$ constructed in Theorem 6.2.3 and the corresponding epimorphism $\phi\colon W_{\mathcal{T}} \to S(\mathcal{T})$. Since $W_{\mathcal{T}}$ is finitely generated by Lemma 6.2.4 and a strongly E^*-unitary inverse semigroup which admits a 0-restricted idempotent-pure pre-homomorphism into the abelian group with zero $(\mathbb{Z}^2)^{\bar{0}}$ by Theorem 6.2.3, from Proposition 6.2.6 and Proposition 1.2.8 we conclude that $W_{\mathcal{T}}/\ker\phi$ is infinitely presented as a strongly E^*-unitary inverse semigroup which admits a 0-restricted idempotent-pure pre-homomorphism into an abelian group with zero. Since, from Proposition 6.2.5, $S(\mathcal{T}) \simeq W_{\mathcal{T}}/\ker\phi$, we have our conclusion. $\square$

Remark 6.2.8. It is not hard to see that the construction developed in this section does not work for $n = 1$ but that it can be generalized to an n-dimensional hypercubic tiling semigroup $S(\mathcal{T})$, with $n \geqslant 3$. In fact, taking, in Definition 6.2.2, $\Gamma(X;C)$ to be the Cayley graph of the usual group presentation for $\mathbb{Z}^n$,

$$Gp\,\langle a_1,\ldots,a_n \mid a_i + a_j = a_j + a_i \ (i,j \in [n])\rangle$$

(so that, in particular, $X = \{a_1,\ldots,a_n\}$ has n elements), the notion of Σ-coloured subgraph of $\mathbb{Z}^n$ is generalized to arbitrary dimension. Also, considering the pairs $((\Gamma,\alpha),g)$ with (Γ,α) a Σ-coloured subgraph of $\mathbb{Z}^n$, some translation (from $\mathbb{Z}^n$) of which is a finite connected Σ-coloured subset of $\mathcal{T} = (\mathbb{Z}^n,\tau)$, and $0, g \in V(\Gamma)$, for the non-zero elements and the operation as then defined, the finitely generated strongly E^*-unitary inverse semigroup $W_{\mathcal{T}}$ that admits a 0-restricted idempotent-pure pre-homomorphism into the abelian group with zero $(\mathbb{Z}^n)^{\bar{0}}$ in Theorem 6.2.3 and Lemma 6.2.4 is generalized as well. In addition, the mapping $\phi\colon W_{\mathcal{T}} \to S(\mathcal{T})$ defined on the non-zero elements by $((\Gamma,\alpha),g)\phi = [z,(V(\Gamma),\alpha)+z,g+z]$, with $z \in \mathbb{Z}^n$ such that $(V(\Gamma),\alpha)+z$ is a Σ-coloured subset of $(\mathbb{Z}^2,\tau)$, is an epimorphism (cf. Proposition 6.2.5); the congruence $\ker\phi$ admits the same characterization and the proof of Proposition 6.2.6 can be adapted, simply by taking a and b to be any distinct elements of X, leading to the corresponding conclusion; finally, the n-dimensional analogue to Theorem 6.2.7 follows. Thus, we can conclude that, as in this result for two-dimensional hypercubic tiling semigroups, any n-dimensional hypercubic tiling semigroup $S(\mathcal{T})$ with $n \geqslant 2$ is infinitely presented, even as an E^*-unitary inverse that admits an idempotent-pure pre-homomorphism into an abelian group with zero.

Note how these results contrast with the one-dimensional case. A one-dimensional tiling semigroup is finitely or infinitely presented according to whether or not the set of minimal forbidden words of its language is finite. In particular, a one-dimensional tiling semigroup with empty set of minimal forbidden words is finitely presented. Thus, for one-dimensional tilings, the part of the operation of the tiling semigroup that does not have to do with the matching of colours (or lengths) is always finitely presented — Theorem 6.2.7 and Remark 6.2.8 show that it is not so in higher dimensions, even in the case of hypercubic tilings.

Periodic n-dimensional hypercubic tilings with $\mathbf{n} \geqslant 2$

Recall that a one-dimensional periodic tiling is one whose language consists of all finite factors of words in w^*, for some word w; that is, the tiling is obtained by the repetition of some pattern. If we think of such a tiling as a pair $(\mathbb{Z},\tau)$, this means that there exists a vector in $\mathbb{Z}$, namely $v = |w|$, where $|w|$ is the length of the word w, such that, for all $x \in \mathbb{Z}$, we have $\tau(x) = \tau(x+v)$. Thus,

Definition 6.2.9. 1. We say that an n-dimensional hypercubic tiling $\mathcal{T} = (\mathbb{Z}^n,\tau)$ is *periodic* if there exist n linearly independent vectors $v_1,\ldots,v_n$ of $\mathbb{Z}^n$ such that, for all $x \in \mathbb{Z}^n$ and for all v_i with $i \in [n]$, we have $\tau(x) = \tau(x+v_i)$.

2. Let $\mathcal{T} = (\mathbb{Z}^n,\tau)$ be a periodic n-dimensional hypercubic tiling over Σ. We say that a finite connected Σ-coloured subset P of $(\mathbb{Z}^n,\tau)$ is a *period* of $\mathcal{T}$ if

$$(\mathbb{Z}^n,\tau) = \bigcup_{j_1,\ldots,j_n\in\mathbb{Z}} \big((P,\tau|_P) + j_1v_1 + \ldots + j_nv_n\big).$$

So that there is no unnecessary overlap in the tiling, one usually assumes that $P\cap(P+v_i)$ and $(P+v_i)\cap(P+v_j)$ are disjoint sets for all $i,j \in [n]$ with $i \neq j$.

Example 6.2.10. Consider the following two-dimensional periodic tiling $\mathcal{T}$ over $\Sigma = \{a, b, c\}$:

$$
\begin{array}{ccc}
 & \vdots & \\
\cdots & \begin{array}{|c|c|c|c|c|c|c|}
\hline
a & a & a & a & a & a & a \\ \hline
b & c & b & c & b & c & b \\ \hline
a & a & a & a & a & a & a \\ \hline
b & c & b & c & b & c & b \\ \hline
a & a & a & a & a & a & a \\ \hline
\end{array} & \cdots \\
 & \vdots &
\end{array}
$$

It is easy to see that $\mathcal{T}$ is periodic and that no period of $\mathcal{T}$ can have less than four tiles. The following are examples of four-tile periods of $\mathcal{T}$:

$$
\begin{array}{|c|c|}
\hline
a & a \\ \hline
b & c \\ \hline
\end{array}
\qquad
\begin{array}{|c|c|}
\hline
a & a \\ \hline
c & b \\ \hline
\end{array}
\qquad
\begin{array}{cc}
 & c \\
a & a \\
b &
\end{array}
\qquad
\begin{array}{cc}
 & b \\
a & a \\
c &
\end{array}
$$

For the vectors v_1 and v_2 in the definition of periodic tiling, one could take $v_1 = (2, 0)$ and $v_2 = (0, 2)$ or $v_1 = (2, 0)$ and $v_2 = (2, 2)$, for instance.

Clearly, the following pattern is also a period of the tiling

$$
\begin{array}{|c|c|c|c|}
\hline
a & a & a & a \\ \hline
b & c & b & c \\ \hline
\end{array}
$$

for which the choice of vectors could be $v_1 = (4, 0)$ and $v_2 = (0, 2)$.

In Section 6.1, we showed that every one-dimensional periodic tiling semigroup is finitely presented (as an inverse semigroup), since the set of minimal forbidden words of the tiling language of such a tiling is always finite. From what we have seen in the previous section, that cannot be the case for a periodic n-dimensional hypercubic tiling semigroup; the question one could ask, thus, is whether a periodic n-dimensional hypercubic tiling semigroup with $n \geqslant 2$ and colours in a set Σ requires a finite or infinite number of additional relations beyond those which are necessary to define an n-dimensional hypercubic tiling semigroup with colours in Σ in which every combination of colours exists, that is, the free n-dimensional hypercubic tiling semigroup over Σ. In this way, we will be investigating whether or not it is possible to encode in a finite way the information concerning the arrangement of colours in a periodic tiling.

Recall from Chapter 4 that an arbitrary n-dimensional hypercubic tiling semigroup $S(\mathcal{T})$ is always isomorphic to the Rees factor semigroup $F(n, \Sigma)/I(L(\mathcal{T}))$ of the free n-dimensional hypercubic tiling semigroup over Σ by its ideal

$$I(L(\mathcal{T})) = \{(p, P, q) \in F(n, \Sigma) \colon P \notin L(\mathcal{T})\} \cup \{0\}$$

(cf. Proposition 4.2.8). Proposition 1.2.13 from Chapter 1 justifies the following terminology.

Definition 6.2.11. Let $\mathcal{T}$ be an n-dimensional hypercubic tiling over an alphabet Σ. We say that $S(\mathcal{T})$ is *finitely presented as a tiling semigroup* if $I(L(\mathcal{T}))$ is a finitely generated ideal of $F(n,\Sigma)$ and that it is *infinitely presented as a tiling semigroup* otherwise.

To play the role of the minimal forbidden words in the one-dimensional case, given an n-dimensional hypercubic tiling $\mathcal{T}$ over a set Σ, we consider the set

$$\begin{aligned} M_\Sigma(L(\mathcal{T})) = \{\,(Q,\beta)\colon\ & (Q,\beta) \text{ is a finite connected } \Sigma\text{-coloured subset of } \mathbb{Z}^n, \\ & Q_0 = 0,\ (Q,\beta) \notin L(\mathcal{T}) \text{ and } (T,\beta|_T) - t_0 \in L(\mathcal{T}) \\ & \text{for each proper connected non-empty subset } T \text{ of } Q\,\}\ . \end{aligned}$$

Just as for one-dimensional tilings, we have

Proposition 6.2.12. *Let $\mathcal{T}$ be an n-dimensional hypercubic tiling over a set Σ. Then $S(\mathcal{T})$ is finitely presented as a tiling semigroup if and only if $M_\Sigma(L(\mathcal{T}))$ is finite.*

The following result will be useful:

Lemma 6.2.13. *Let K be an order ideal of a partially ordered set X such that, for each $x \in X$, the filter $Y_x = \{y \in X\colon x \leqslant y\}$ generated by x is finite. Then K is finitely generated if and only if the set M of maximal elements of K,*

$$M = \{x \in K\colon \forall\, y \in X,\ x < y \Rightarrow y \notin K\}\,,$$

is finite.

Proof. Suppose K is finitely generated, and let A be a finite generating set for K. We claim that $M \subseteq A$. So let $x \in M$. Then $x \in K$, so that $x \leqslant a$ for some $a \in A$. If $x < a$, then, by definition of M, we have $a \notin K$, a contradiction. Thus, $x = a$, and so $x \in A$, as claimed. Therefore, since A is finite, so is M.

Conversely, suppose M is finite. We will show that K is generated by M as an order ideal. Let $x \in K$. If $x \in M$, we have our claim. Now assume that $x \notin M$. Then there exists $y_1 \in X$ such that $x < y_1$ and $y_1 \in K$. If $y_1 \in M$, we have our conclusion; if not, then there exists $y_2 \in X$ such that $x < y_1 < y_2$ and $y_2 \in K$. Again, if $y_2 \in M$ we have our conclusion and, otherwise, there exists $y_3 \in X$ such that $x < y_1 < y_2 < y_3$ and $y_3 \in K$. Since $y_1, y_2, y_3, \ldots$ belong to $Y_x = \{y \in X\colon x \leqslant y\}$, which is finite by assumption, there exists $z \in Y_x$ such that $z \in K$ and $z < w$ implies that $w \notin K$, for each $w \in X$. But then $z \in M$, and so $x \leqslant z$ with $z \in M$. Hence, K is generated by M as an order ideal. ☐

Proof of Proposition 6.2.12. By definition, $S(\mathcal{T})$ is finitely presented as a tiling semigroup if and only if the ideal $I(L(\mathcal{T}))$ of the free n-dimensional tiling semigroup $F = F(n,\Sigma)$ over Σ is finitely generated. Since the lattice of ideals of a semigroup S is isomorphic to the lattice of order ideals of the partially ordered set $S/\mathcal{J}$ (via the isomorphism that takes an ideal I to the order ideal $I/\mathcal{J}$), we have that $I(L(\mathcal{T}))$ is finitely generated if and only if $I(L(\mathcal{T}))/\mathcal{J}_F$ is finitely generated.

In [32], Lawson showed that non-zero elements of a tiling semigroup are $\mathcal{D}$-related if and only if their underlying patterns are equivalent. On the other hand, Zhu proved in [61] that the Green's relations $\mathcal{D}$ and $\mathcal{J}$ coincide in any tiling semigroup. Thus, it is clear that $(p,P,q)\mathcal{J}_F(r,R,s)$ if and only if $P=R$ and, therefore, $F/\mathcal{J}_F$ is in a bijective correspondence with the tiling language $L(F)$ of F. Notice that $(p,P,q) \leqslant_{\mathcal{J}_F} (r,R,s)$ if and only if $R+u \subseteq P$ for some $u \in \mathbb{Z}^n$, as direct calculations readily show. It follows that, considering on $L(F)$ the order $\leqslant$ defined by: for all $P,Q \in L(F)$,

$$\begin{aligned} P \leqslant Q \quad (\text{in } L(F)) &\Leftrightarrow (0,P,0) \leqslant (0,Q,0) \quad (\text{in } F) \\ &\Leftrightarrow \exists\, u \in \mathbb{Z}^n \text{ such that } Q+u \leqslant P\,, \end{aligned}$$

we have that the partially ordered sets $F/\mathcal{J}_F$ and $L(F)$, with their respective orders, are isomorphic. (Recall that "$Q+u \leqslant P$" means that $Q+u \subseteq P$ as well as that the colours agree.) Consequently, we can identify $F/\mathcal{J}_F$ and $L(F)$ and the order ideal $I(L(\mathcal{T}))/\mathcal{J}_F$ of $F/\mathcal{J}_F$ with the order ideal $K(\mathcal{T}) = \{P \in L(F)\colon P \notin L(\mathcal{T})\} = L(F) \setminus L(\mathcal{T})$ of $L(F)$. Hence, $I(L(\mathcal{T}))$ is finitely generated if and only if $K(\mathcal{T})$ is finitely generated.

Now, since

$$\begin{aligned} M &= \{P \in K(\mathcal{T})\colon \forall\, Q \in L(F),\ P < Q \Rightarrow Q \notin K(\mathcal{T})\} = \\ &= \{P \notin L(\mathcal{T})\colon \forall\, Q \in L(F),\ P < Q \Rightarrow Q \in L(\mathcal{T})\} = M_\Sigma(\mathcal{T})\,, \end{aligned}$$

by definition of $M_\Sigma(\mathcal{T})$, in view the previous lemma we need only check that, for each $P \in L(F)$, the filter $\{A \in L(F)\colon P \leqslant A\}$ generated by P is finite; this shows that $K(\mathcal{T})$ is finitely generated if and only if $M_\Sigma(\mathcal{T})$ is finite. In fact, it is clear that, for each $P \in L(F)$, there are only finitely many $A \in L(F)$ — that is, (finite and connected) Σ-coloured subsets of $\mathbb{Z}^n$ with minimum element 0 — with $P \leqslant A$ — that is, with $A+u \subseteq P$ and agreeing colouring maps for some $u \in \mathbb{Z}^n$.

Hence, $I(L(\mathcal{T}))$ is finitely generated if and only if $M_\Sigma(L(\mathcal{T}))$ is finite, or, equivalently, $S(\mathcal{T})$ is finitely presented as a tiling semigroup if and only if $M_\Sigma(L(\mathcal{T}))$ is finite. $\square$

Example 6.2.10 gives a negative answer to the question asked: is the tiling semigroup of any periodic n-dimensional hypercubic tiling $\mathcal{T}$ with $n \geqslant 2$ finitely presented as a tiling semigroup? In fact, it is easy to check that, for each $m \geqslant 1$, the Σ-coloured subset (Q_m, β) of $\mathbb{Z}^2$

$$\begin{array}{cccccc} b & & & & & \\ a & a & a & \cdots & a & a \\ & & & & & b \end{array}$$

with minimum element 0 and such that the string of a's has length $2m$, belongs to $M_\Sigma(L(\mathcal{T}))$, so that, by the previous lemma, $S(\mathcal{T})$ is infinitely presented, even as a tiling semigroup.

Of course, this need not be always the case.

Proposition 6.2.14. *Let $\mathcal{T} = (\mathbb{Z}^n, \tau)$ be an n-dimensional periodic hypercubic tiling. If $\mathcal{T}$ admits a period in which all tiles have distinct colours, then $S(\mathcal{T})$ is finitely presented as a tiling semigroup.*

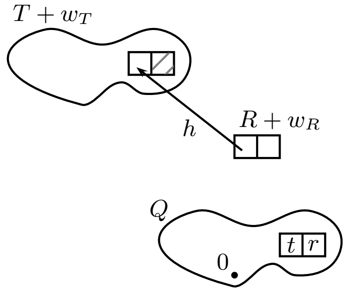
Proof. By assumption,

$$(\mathbb{Z}^n, \tau) = \bigcup_{j_1,\ldots,j_n \in \mathbb{Z}} \big((P, \tau|_P) + j_1 v_1 + \ldots + j_n v_n\big) \tag{6.5}$$

for some period P in which no two distinct tiles have the same colour and linearly independent vectors $v_1, \ldots, v_n \in \mathbb{Z}^n$. Let H denote the subgroup generated by $v_1, \ldots, v_n$. Then, for each $x \in \mathbb{Z}^n$ and $h \in H$, we have $\tau(x + h) = \tau(x)$. In fact, the converse is also true: if $\tau(x) = \tau(y)$, then $x - y \in H$, for all $x, y \in \mathbb{Z}^n$. To see this, note that, from (6.5), there exist $h_1, h_2 \in H$ such that $x + h_1, y + h_2 \in P$. But then $\tau(x + h_1) = \tau(x) = \tau(y) = \tau(y + h_2)$. By the hypothesis that distinct vertices in P have distinct colours, this implies that $x + h_1 = y + h_2$. Thus $x - y = h_2 - h_1 \in H$.

In view of Proposition 6.2.12, it suffices to show that $M_\Sigma(L(\mathcal{T}))$ is a finite set. We claim that $M_\Sigma(L(\mathcal{T}))$ consists only of two-vertex elements; since we only consider finite type tilings, this yields the desired conclusion. In fact, let (Q, β) be a finite connected Σ-coloured subset of $\mathbb{Z}^n$ with minimum element 0 such that $|Q| \geqslant 3$ and $(T, \beta|_T) - t_0 \in L(\mathcal{T})$ for each proper connected subset T of Q; we will show that $(Q, \beta) \in L(\mathcal{T})$.

Since Q is finite, connected, and has at least three elements, there exist elements $t, r \in Q$ such that $T = Q \setminus \{r\}$ and $R = \{t, r\}$ are connected subsets of Q (take T to be the vertex set of a maximal connected subgraph of the full connected graph with vertex set Q). Therefore, $(T, \beta|_T)$ and $(R, \beta|_R)$ are proper Σ-coloured connected subsets of (Q, β). It follows that $(T, \beta|_T) - t_0$ and $(R, \beta|_R) - r_0$ belong to $L(\mathcal{T})$, or, equivalently, that $(T, \beta|_T) + w_T$ and $(R, \beta|_R) + w_R$ are Σ-coloured connected subsets of $(\mathbb{Z}^n, \tau)$ for some $w_T, w_R \in \mathbb{Z}^n$.



Since

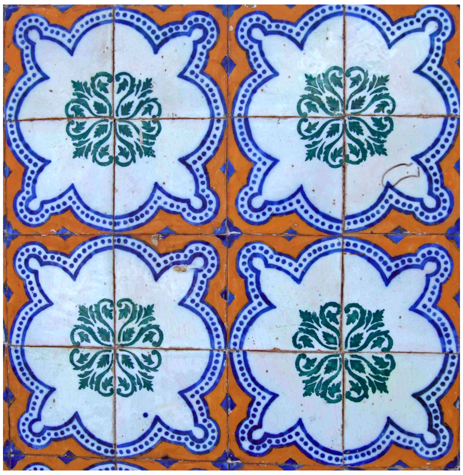
$$\tau(t + w_T) = \beta|_T(t) = \beta(t) = \beta|_R(t) = \tau(t + w_R)\,,$$

by the first paragraph $w_T - w_R \in H$, or, equivalently, $w_T = w_R + h$ for some $h \in H$. Thus, $(R, \beta|_R) + w_R + h$ is a Σ-coloured connected subset of $(\mathbb{Z}^n, \tau)$ by definition of periodic tiling. Since

$$(Q, \beta) + w_T = \big((T, \beta|_T) + w_T\big) \cup \big((R, \beta|_R) + w_T\big) = \big((T, \beta|_T) + w_T\big) \cup \big((R, \beta|_R) + w_R + h\big)\,,$$

we conclude that $(Q, \beta) + w_T$ is a Σ-coloured connected subset of $(\mathbb{Z}^n, \tau)$. Hence, $(Q, \beta) \in L(\mathcal{T})$, and so $(Q, \beta) \notin M_\Sigma(L(\mathcal{T}))$. It follows that no element of $M_\Sigma(L(\mathcal{T}))$ has more than two vertices. $\square$

As a final remark, we note that this implies that the tiling on the next page[1], familiar to the residents of Lisbon, has finitely presented tiling semigroup.



[1]Photograph by Donald B. McAlister.

Chapter 7

Isomorphism of hypercubic tilings

As mentioned earlier in Chapter 4, in [25] Kellendonk remarks that two n-dimensional tilings have the same tiling semigroup if and only if each pattern in the first appears as a pattern in the second, and vice-versa, which, in terms of hypercubic tiling semigroups, is equivalent to saying that the semigroups are equal if and only if they have the same tiling languages.

In this chapter, we turn to the question of when two n-dimensional hypercubic tilings have isomorphic tiling semigroups. This question is relevant because it asks what kind of information about the tiling is the tiling semigroup sensitive to. The answer is that, under mild conditions, the semigroups of two n-dimensional hypercubic tilings are isomorphic if and only if their languages are the same, subject to a bijective change of colours and a symmetry of the n-dimensional hypercube. This result highlights the importance of the tiling language as far as tiling semigroups are concerned.

This question was also motivated by a result of Masuda and Morita in [37]. As mentioned in Section 3.3, in [37], the authors considered a construction consisting of a bialgebra associated to the tiling semigroup and investigated under which conditions one-dimensional tilings give rise to isomorphic associated bialgebras. Here, we tried to find a clearer and simpler answer to a more general problem.

We begin this chapter by presenting the group of symmetries of the n-dimensional hypercube, known as the *full symmetry group of the n-dimensional hypercube*, and by stating some facts that will be useful in the present chapter. Although this is an interesting group, it is not easy to find more than a couple of lines about it in the literature. Reference [18] is an exception, providing some deep information on the group and pointing to many connections to other areas of Mathematics, but with few proofs. For that reason, an appendix on this group consisting of some extended proofs of facts about G_n has been added at the end of this thesis.

After establishing the main result of this section, Theorem 7.2.1, which characterizes n-dimensional hypercubic tilings whose tiling semigroups are isomorphic, and for which a few somewhat technical lemmas are proved in advance, we turn to the one-dimensional case, where a simple characterization of one-dimensional isomorphic tiling semigroups can be given in terms of their languages.

As in Section 6.2 of the previous chapter, we did not find it worthy to present our proofs in the more general framework of the inverse semigroup associated with an n-dimensional factorial language. However, generalizations will be mentioned when appropriate and a comparison between one-dimensional tiling semigroups and inverse semigroups associated with (one-dimensional) factorial languages will take place in the final section.

7.1 The full symmetry group of the n-dimensional hypercube

For convenience, in this section we regard the n-dimensional hypercube as the set $[-1,1]^n$, rather than $[0,1]^n$ as in the definition of hypercubic tiling given in Section 3.4. The two conventions are, of course, equivalent.

The n-tuples of the n-dimensional hypercube in which each component equals either 1 or -1 are called *vertices* of the hypercube. We say that two vertices are *neighbours* if they differ in a single component. Thus, u and v are neighbours if and only if $u-v$ has exactly one non-zero component, which is either 2 or -2. Of course, each vertex has n neighbours.

A *symmetry* of the n-dimensional hypercube is a permutation of its vertices that preserves neighbours.

Example 7.1.1. In $\mathbb{R}^2$, such a symmetry is either the identity, a reflection about the x-axis, y-axis, the straight lines $y=x$ or the straight line $y=-x$, or a rotation about the origin of $\frac{\pi}{2}$, or π or $\frac{3\pi}{2}$. In Figure 7.1, we have a representation of the symmetries of the two-dimensional hypercube, which were denoted by id, σ_x, σ_y, $\sigma_{y=x}$, $\sigma_{y=-x}$, $\sigma_{\frac{\pi}{2}}$, σ_π, and $\sigma_{\frac{3\pi}{2}}$, respectively.

The group consisting of all symmetries of the n-dimensional hypercube under composition is the *full symmetry group of the n-dimensional hypercube* and it is denoted by G_n.

In the Appendix, we will show that each symmetry of the n-dimensional hypercube extends naturally to an automorphism $\tilde{\sigma}$ of $\mathbb{Z}^n$. As a consequence, the elements of G_n are linear maps. Moreover, the matrix of the automorphism that extends a symmetry σ, with respect to the standard basis $\{u_1,\ldots,u_n\}$ of $\mathbb{Z}^n$,

$$M_{\tilde{\sigma}} = \begin{pmatrix} u_1\sigma \\ \vdots \\ u_n\sigma \end{pmatrix},$$

has exactly one non-zero entry in each column and in each row, which is either 1 or -1, that is, is a *signed permutation matrix*. Since $\tilde{\sigma}$ extends σ, the matrix $M_{\tilde{\sigma}}$ also represents σ. In the notation of the previous example, we have

$$M_{id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad M_{\sigma_x} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad M_{\sigma_y} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$M_{\sigma_{y=x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad M_{\sigma_{y=-x}} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

$$M_{\sigma_{\frac{\pi}{2}}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad M_{\sigma_\pi} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad M_{\sigma_{\frac{3\pi}{2}}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We will also show that, conversely, every $n \times n$ signed permutation matrix is the matrix of some automorphism of $\mathbb{Z}^n$ which restricts to a symmetry of the n-dimensional hypercube (cf. Appendix, Proposition 3).

Since any symmetry from G_n takes $\mathbb{Z}^n$ onto itself, there is a natural action of G_n on the set of Σ-coloured subsets of $\mathbb{Z}^n$: if σ is a symmetry from G_n and (A, α) is a Σ-coloured subset of $\mathbb{Z}^n$, then $(A, \alpha)\sigma = (B, \beta)$ is the Σ-coloured subset of $\mathbb{Z}^n$ with $B = \{x\sigma : x \in A\}$ and $\beta : B \to \Sigma$ the map defined by $\beta(b) = \alpha(b\sigma^{-1})$, for all $b \in B$. Thus, the action of σ may change the position of a vertex, but it does not change the colour of the corresponding vertex. Also note that if (A, α) is finite, non-empty, or connected, then so is $(A, \alpha)\sigma$.

7.2 Main theorem

In what follows, we will use the original representation of a tiling semigroup, with the identification made in Section 3.4 between patterns in a hypercubic tiling $\mathcal{T}$ over Σ and finite, connected Σ-coloured subsets of the Σ-coloured subset $(\mathbb{Z}^n, \tau)$ associated with $\mathcal{T}$. By the end of this subsection, however, we will go back to the language representation of $S(\mathcal{T})$.

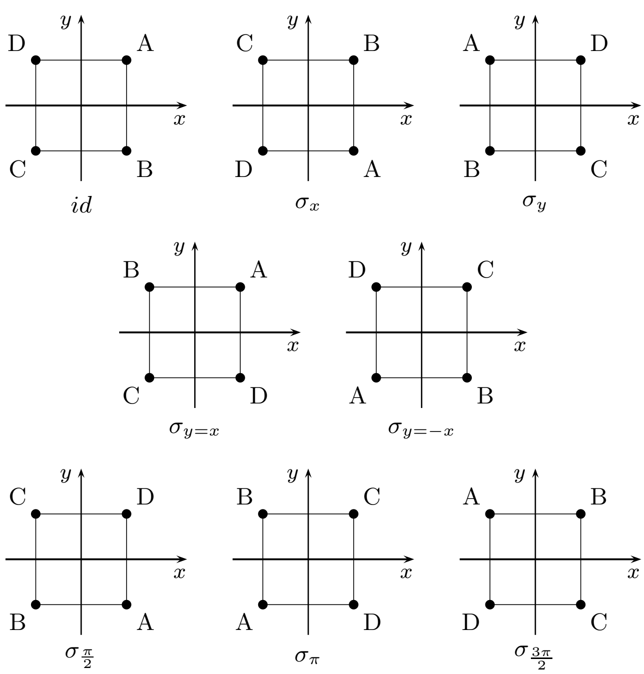


Figure 7.1: The elements of G_2

Given two finite alphabets Σ_1 and Σ_2 and a bijection $\varphi\colon \Sigma_1 \to \Sigma_2$, denote by φ^* the map that naturally extends φ to the Σ_1-coloured subsets of $\mathbb{Z}^n$: for each Σ_1-coloured subset (A,α) of $\mathbb{Z}^n$, set $(A,\alpha)\varphi^*$ to be the Σ_2-coloured subset (A,β) of $\mathbb{Z}^n$ with $\beta(a) = \varphi(\alpha(a))$, for all $a \in A$. As usual, whenever the colour maps are well understood by the context, we omit any mention of them, and simply write $A\varphi^*$.

The main theorem of this section is the following:

Theorem 7.2.1. *Let $\mathcal{T}_1 = (\mathbb{Z}^n, \tau_1)$ and $\mathcal{T}_2 = (\mathbb{Z}^n, \tau_2)$ be n-dimensional hypercubic tilings with $\tau_1\colon \mathbb{Z}^n \to \Sigma_1$ and $\tau_2\colon \mathbb{Z}^n \to \Sigma_2$. Let $\varphi\colon \Sigma_1 \to \Sigma_2$ be a bijection and σ a symmetry of G_n. Suppose that if A is a finite, connected Σ_1-coloured subset of $(\mathbb{Z}^n, \tau_1)$ then $A\varphi^*\sigma + v$ is a finite, connected Σ_2-coloured subset of $(\mathbb{Z}^n, \tau_2)$ for some $v \in \mathbb{Z}^n$, and that if B is a finite, connected Σ_2-coloured subset of $(\mathbb{Z}^n, \tau_2)$ then $B(\varphi^{-1})^*\sigma^{-1} + v$ is a finite, connected Σ_1-coloured subset of $(\mathbb{Z}^n, \tau_1)$ for some $v \in \mathbb{Z}^n$. Then the mapping $\theta\colon S(\mathcal{T}_1) \to S(\mathcal{T}_2)$ defined by:*

$$\begin{cases} 0\theta = 0, \\ [a, A, b]\theta = [a\varphi\sigma + v, A\varphi^*\sigma + v, b\varphi\sigma + v], \quad [a, A, b] \in S(\mathcal{T}_1) \setminus \{0\}\ , \end{cases}$$

where $v \in \mathbb{Z}^n$ is such that $A\varphi^*\sigma + v$ is a finite, connected Σ_2-coloured subset of $(\mathbb{Z}^n, \tau_2)$, is an isomorphism. Conversely, if both $(\mathbb{Z}^n, \tau_1)$ and $(\mathbb{Z}^n, \tau_2)$ contain all finite, connected coloured subsets up to four elements, then every isomorphism from $S(\mathcal{T}_1)$ onto $S(\mathcal{T}_2)$ is of this form for some bijection $\varphi\colon \Sigma_1 \to \Sigma_2$ and some symmetry $\sigma \in G_n$.

In order to prove this result, we begin by establishing a few helpful facts about the relation between the images of the generators when the semigroups of two n-dimensional hypercubic tilings are isomorphic. Although the results are very natural, their proofs are rather long and technical.

Recall from Proposition 3.4.8 that the tiling semigroup $S(\mathcal{T})$ of an n-dimensional hypercubic tiling $\mathcal{T} = (\mathbb{Z}^n, \tau)$ with colours from a set Σ, is generated, as an inverse semigroup with zero, by the $|\Sigma|$ single-tile idempotents, which we will usually denote by e_i (with $i \in \Sigma$) and the following two-tile elements $a_{1(i,j)}, a_{2(i,j)}, \ldots$ and $a_{n(i,j)}$ (with $i, j \in \Sigma$): for each $k \in [n]$, $a_{k(i,j)} = [a, (A,\alpha), b]$ with $A = \{a, b\}$, (A,α) is a Σ-coloured subset of $(\mathbb{Z}^n, \tau)$, $\alpha(a) = i$, $\alpha(b) = j$ and $b - a$ is the k^{th} standard basis vector of $\mathbb{Z}^n$.

In Lemmas 7.2.2, 7.2.3, and 7.2.4, let $\mathcal{T}_1 = (\mathbb{Z}^n, \tau_1)$ and $\mathcal{T}_2 = (\mathbb{Z}^n, \tau_2)$ be n-dimensional hypercubic tilings, suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ contain all finite, connected coloured subsets with four tiles (and, therefore, all finite, connected coloured subsets with one, two or three tiles), and that $\theta\colon S(\mathcal{T}_1) \to S(\mathcal{T}_2)$ is an isomorphism. For clarity, we will denote the generators of $S(\mathcal{T}_1)$ by e_i, with $i \in \Sigma_1$, and $a_{k(i,j)}$, with $k \in [n]$ and $i, j \in \Sigma_1$, and the generators of $S(\mathcal{T}_2)$ by f_i, with $i \in \Sigma_2$, and $b_{k(i,j)}$, with $k \in [n]$ and $i, j \in \Sigma_2$.

Two well-known results about tiling semigroups will be of use; both can be found in [58]. The first one concerns the natural partial order in an arbitrary tiling semigroup $S(\mathcal{T})$ and was already mentioned in Section 5.1: for all $[a, A, b], [c, C, d] \in S(\mathcal{T})$, we have $[a, A, b] \leqslant [c, C, d]$

if and only if $C + x \subseteq A$ for some $x \in \mathbb{R}^n$ such that $c + x = a$ and $d + x = b$. In particular, the single-tile idempotents are the maximal idempotents. Since isomorphisms map maximal idempotents onto maximal idempotents, then for all $i \in \Sigma_1$ we have $e_i\theta = f_j$, for some $j \in \Sigma_2$.[1] Consider the mapping $\varphi\colon \Sigma_1 \to \Sigma_2$ defined by $\varphi(i) = j$ if and only if $e_i\theta = f_j$, for all $i \in \Sigma_1$. It is trivial to check, using the fact that θ and θ^{-1} are bijections, that φ is a bijection as well.

The second result concerns the Green relation $\mathcal{D}$, more precisely it says that, for all $[a, A, b], [c, C, d] \in S(\mathcal{T})$, we have $[a, A, b]\mathcal{D}[c, C, d]$ if and only if $A = C + x$ for some $x \in \mathbb{R}^n$.

Lemma 7.2.2. *For all $a_{k(i,j)} \in S(\mathcal{T}_1)$, there exists $m \in [n]$ for which*

$$a_{k(i,j)}\theta = b_{m(\varphi(i),\varphi(j))} \qquad \text{or} \qquad a_{k(i,j)}\theta = b^{-1}_{m(\varphi(j),\varphi(i))}\,.$$

Proof. Since isomorphisms map $\mathcal{D}$-classes onto $\mathcal{D}$-classes, from the description of $\mathcal{D}$ in tiling semigroups we have that for each two-tile element $a_{k(i,j)} \in S(\mathcal{T}_1)$ there exists a two-tile element $b_{m(r,s)} \in S(\mathcal{T}_2)$ such that $D_{a_{k(i,j)}}\theta = D_{b_{m(r,s)}}$, that is,

$$\left\{a_{k(i,j)}, a^{-1}_{k(i,j)}, a_{k(i,j)}a^{-1}_{k(i,j)}, a^{-1}_{k(i,j)}a_{k(i,j)}\right\}\theta = \left\{b_{m(r,s)}, b^{-1}_{m(r,s)}, b_{m(r,s)}b^{-1}_{m(r,s)}, b^{-1}_{m(r,s)}b_{m(r,s)}\right\}\,.$$

Since θ maps idempotents to idempotents, we have $a_{k(i,j)}\theta = b_{m(r,s)}$ or $a_{k(i,j)}\theta = b^{-1}_{m(r,s)}$. More precisely, we must have $a_{k(i,j)}\theta = b_{m(\varphi(i),\varphi(j))}$ or $a_{k(i,j)}\theta = b^{-1}_{m(\varphi(j),\varphi(i))}$. In fact, if $a_{k(i,j)}\theta = b_{m(r,s)}$, then $a_{k(i,j)} = e_i\, a_{k(i,j)}$ implies that

$$a_{k(i,j)}\theta = \left(e_i\, a_{k(i,j)}\right)\theta = e_i\theta\, a_{k(i,j)}\theta \;\Leftrightarrow\; b_{m(r,s)} = f_{\varphi(i)}\, b_{m(r,s)} \neq 0\,,$$

which in turn implies that $r = \varphi(i)$. From the fact that $a_{k(i,j)} = a_{k(i,j)}\, e_j$ we conclude, similarly, that $s = \varphi(j)$. Thus, in this case, $a_{k(i,j)}\theta = b_{m(\varphi(i),\varphi(j))}$. If $a_{k(i,j)}\theta = b^{-1}_{m(r,s)}$, then $a_{k(i,j)} = e_i\, a_{k(i,j)}$ now implies that

$$a_{k(i,j)}\theta = \left(e_i\, a_{k(i,j)}\right)\theta = e_i\theta\, a_{k(i,j)}\theta \;\Leftrightarrow\; b^{-1}_{m(r,s)} = f_{\varphi(i)}\, b^{-1}_{m(r,s)} \neq 0\,,$$

and so $s = \varphi(i)$. Similarly, $r = \varphi(j)$ and therefore $a_{k(i,j)}\theta = b^{-1}_{m(\varphi(j),\varphi(i))}$. $\square$

The following result concerns only one-dimensional (hypercubic) tilings. For simplicity of notation, here we denote the two-tile generators of $S(\mathcal{T}_1)$ (respectively, $S(\mathcal{T}_2)$) by $a_{(i,j)}$, with $i, j \in \Sigma_1$ (respectively, $b_{(i,j)}$, with $i, j \in \Sigma_2$).

Lemma 7.2.3. *If $a_{(i,j)}\theta = b_{(\varphi(i),\varphi(j))}$ for some $a_{(i,j)} \in S(\mathcal{T}_1)$, then $a_{(r,s)}\theta = b_{(\varphi(r),\varphi(s))}$, for all $a_{(r,s)} \in S(\mathcal{T}_1)$; if $a_{(i,j)}\theta = b^{-1}_{(\varphi(j),\varphi(i))}$ for some $a_{(i,j)} \in S(\mathcal{T}_1)$, then $a_{(r,s)}\theta = b^{-1}_{(\varphi(s),\varphi(r))}$, for all $a_{(r,s)} \in S(\mathcal{T}_1)$.*

Proof. Without loss of generality, we will assume that $\Sigma_1 = \Sigma_2$ and $\varphi = id$. This way, $e_k\theta = f_k$ for all $k \in \Sigma_1 = \Sigma_2$ and Lemma 7.2.2 states that, for all $a_{(i,j)} \in S(\mathcal{T}_1)$, we have $a_{(i,j)}\theta = b_{(i,j)}$ or $a_{(i,j)}\theta = b^{-1}_{(j,i)}$.

[1] This observation was first made by Nick Gilbert.

Let $a_{(i,j)}, a_{(r,s)} \in S(\mathcal{T}_1)$. Then both words ij and rs occur in the tiling language of $\mathcal{T}_1$, and so there exists a word $u \in L(\mathcal{T}_1)$ which has both ij and rs as factors (cf. property (3.1) in Section 3.2). Without loss of generality, suppose that the factor ij occurs to the left of rs. Then either $j = r$ and $u = ijs$ or $u = ijt_1t_2\dots t_mrs$, with $t_1, t_2, \dots, t_m \in \Sigma_1 \cup \{\epsilon\}$. Let $x \in S(\mathcal{T}_1)$ be the element with underlying word u, in-tile in the first letter and out-tile in the last. (In the notation of Example 6.1.2, $x = u\tau$.)

Suppose $u = ijs$. Then $x\theta = \left(a_{(i,j)}\, a_{(j,s)}\right)\theta = a_{(i,j)}\theta\, a_{(j,s)}\theta$ is a non-zero element of $S(\mathcal{T}_2)$. Suppose $a_{(i,j)}\theta = b_{(i,j)}$. If $a_{(j,s)}\theta = b^{-1}_{(s,j)}$, then the fact that the product $x\theta = b_{(i,j)}b^{-1}_{(s,j)}$ is non-zero would imply that $s = i$ and that $x\theta$ was a (two-tile) idempotent (with underlying word ij and both in-tile and out-tile in the first tile), whereas x is a non-idempotent (three-tile) element of $S(\mathcal{T}_1)$, a contradiction. Thus, $a_{(j,s)}\theta = b_{(j,s)}$, that is, $a_{(r,s)}\theta = b_{(r,s)}$. Similarly, if $a_{(i,j)}\theta = b^{-1}_{(j,i)}$ and $a_{(j,s)}\theta = b_{(j,s)}$, then again $s = i$ and the image of the non-idempotent (three-tile) element x would be the (two-tile) idempotent $x\theta$ (which has underlying word ji and both in-tile and out-tile in the second tile). So $a_{(r,s)}\theta = b^{-1}_{(s,r)}$.

Now suppose $u = ijt_1t_2\dots t_mrs$. Then

$$x\theta = \left(a_{(i,j)}\, a_{(j,t_1)} \dots a_{(t_m,r)}\, a_{(r,s)}\right)\theta = a_{(i,j)}\theta\, a_{(j,t_1)}\theta \dots a_{(t_m,r)}\theta\, a_{(r,s)}\theta\,,$$

is a non-zero element of $S(\mathcal{T}_2)$. In particular every product $a_{(i,j)}\theta\, a_{(j,t_1)}\theta, a_{(j,t_1)}\theta\, a_{(t_1,t_2)}\theta, \dots$, and $a_{(t_m,r)}\theta\, a_{(r,s)}\theta$ is non-zero. Using the same arguments as before, we conclude that if $a_{(i,j)}\theta = b_{(i,j)}$, then $a_{(j,t_1)}\theta = b_{(j,t_1)}$, and that if $a_{(i,j)}\theta = b^{-1}_{(j,i)}$, then $a_{(j,t_1)}\theta = b^{-1}_{(t_1,j)}$. Proceeding in a similar way with the remaining pairs we conclude that if $a_{(i,j)}\theta = b_{(i,j)}$, then $a_{(r,s)}\theta = b_{(r,s)}$, and that if $a_{(i,j)}\theta = b^{-1}_{(j,i)}$, then $a_{(r,s)}\theta = b^{-1}_{(s,r)}$, as claimed. $\square$

The following lemma generalizes to n-dimensional tilings, with $n \geqslant 2$, the previous result; that is, it establishes that the bijection φ and the image of a single generator $a_{k(i,j)}$ determines the image of all generators $a_{k(r,s)}$. In addition, it also shows how the image of $a_{k(i,j)}$ restricts the image of the generators $a_{k'(r,s)}$ with $k' \neq k$.

Lemma 7.2.4. *(i) If*

$$a_{k(i,j)}\theta = b_{m(\varphi(i),\varphi(j))} \quad \textit{or} \quad a_{k(i,j)}\theta = b^{-1}_{m(\varphi(j),\varphi(i))}$$

then

$$a_{k'(r,s)}\theta \neq b_{m(\varphi(r),\varphi(s))} \quad \textit{and} \quad a_{k'(r,s)}\theta \neq b^{-1}_{m(\varphi(s),\varphi(r))}\,,$$

for all $a_{k'(r,s)} \in S(\mathcal{T}_1)$ with $k' \neq k$.

(ii) If $a_{k(i,j)}\theta = b_{m(\varphi(i),\varphi(j))}$, then, for all $a_{k(r,s)} \in S(\mathcal{T}_1)$, we have $a_{k(r,s)}\theta = b_{m(\varphi(r),\varphi(s))}$; if $a_{k(i,j)}\theta = b^{-1}_{m(\varphi(j),\varphi(i))}$, then, for all $a_{k(r,s)} \in S(\mathcal{T}_1)$, we have $a_{k(r,s)}\theta = b^{-1}_{m(\varphi(s),\varphi(r))}$.

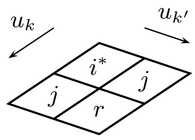
Proof. Again, assume that $\varphi = id$. The outline of the proof is as follows:

1. *if $a_{k(i,j)}\theta = b_{m(i,j)}$ or $a_{k(i,j)}\theta = b^{-1}_{m(j,i)}$, then $a_{k'(j,r)}\theta \neq b_{m(j,r)}$ and $a_{k'(j,r)}\theta \neq b^{-1}_{m(r,j)}$;*

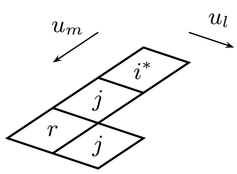
2. *if* $a_{k(i,j)}\theta = b_{m(i,j)}$ *then* $a_{k(r,s)}\theta = b_{m(r,s)}$, *and if* $a_{k(i,j)}\theta = b^{-1}_{m(j,i)}$ *then* $a_{k(r,s)}\theta = b^{-1}_{m(s,r)}$;

3. *if* $a_{k(i,j)}\theta = b_{m(i,j)}$ *or* $a_{k(i,j)}\theta = b^{-1}_{m(j,i)}$, *then* $a_{k'(r,s)}\theta \neq b_{m(r,s)}$ *and* $a_{k'(r,s)}\theta \neq b^{-1}_{m(s,r)}$.

Claim 1. Suppose that $a_{k(i,j)}\theta = b_{m(i,j)}$. Take $a_{k'(j,r)} \in S(\mathcal{T}_1)$ with $k' \neq k$ (which exists since, by assumption, $n \geqslant 2$ and $(\mathbb{Z}^n, \tau_1)$ contains all finite, connected coloured subsets with two tiles). It is easy to see that $a_{k'(j,r)}\theta \neq b^{-1}_{m(r,j)}$. Indeed, the fact that $a_{k(i,j)}a_{k'(j,r)}$ is, by assumption, a non-zero element of $S(\mathcal{T}_1)$ implies that $\left(a_{k(i,j)}a_{k'(j,r)}\right)\theta$ is a non-zero element of $S(\mathcal{T}_2)$, and, if $a_{k'(j,r)}\theta = b^{-1}_{m(r,j)}$, then $\left(a_{k(i,j)}a_{k'(j,r)}\right)\theta = b_{m(i,j)}b^{-1}_{m(r,j)}$ would be a non-zero idempotent, a contradiction since $a_{k(i,j)}a_{k'(j,r)}$ is not an idempotent. Now suppose $a_{k'(j,r)}\theta = b_{m(j,r)}$. Consider the element $x = a_{k(i,j)}a_{k'(j,r)}a^{-1}_{k(j,r)}a^{-1}_{k'(i,j)}$ of $S(\mathcal{T}_1)$. Then x is a non-zero (four-tile) idempotent, whose projection on the $x_kx_{k'}$-plane of $\mathbb{R}^n$ is shown in the figure below



where we denoted by i^* the coinciding in-tile and out-tile of the element. Since θ is injective and $k' \neq k$, $a_{k(j,r)}\theta \neq b_{m(j,r)} = a_{k'(j,r)}\theta$; since $a_{k(i,j)}\theta = b_{m(i,j)}$ and $a_{k(i,j)}a_{k(j,r)}$ is non-zero, then $a_{k(j,r)}\theta \neq b^{-1}_{m(r,j)}$, as otherwise $(a_{k(i,j)}a_{k(j,r)})\theta = b_{m(i,j)}b^{-1}_{m(r,j)}$ would be a (two-tile) idempotent (and $i = r$). Thus, by Lemma 7.2.2, $a_{k(j,r)}\theta = b_{l(j,r)}$ or $a_{k(j,r)}\theta = b^{-1}_{l(r,j)}$, with $l \neq m$. However, because $l \neq m$, the image of x cannot be an idempotent, independently of $a_{k'(i,j)}\theta$: the projection of $a_{k(i,j)}\theta\, a_{k'(j,r)}\theta\, a_{k(j,r)}\theta^{-1}$ on the x_mx_l-plane of $\mathbb{R}^n$ is



(In the figure above, $a_{k(j,r)}\theta = b^{-1}_{l(r,j)}$, but the same conclusion is drawn for $a_{k(j,r)}\theta = b_{l(j,r)}$.) Therefore $a_{k'(j,r)}\theta \neq b_{m(j,r)}$. Analogously, if $a_{k(i,j)}\theta = b^{-1}_{m(j,i)}$, then $a_{k'(j,r)}\theta \neq b_{m(j,r)}$ and $a_{k'(j,r)}\theta \neq b^{-1}_{m(r,j)}$.

Claim 2. Suppose $a_{k(i,j)}\theta = b_{m(i,j)}$ (again, the case $a_{k(i,j)}\theta = b^{-1}_{m(j,i)}$ is analogous). We first show that $a_{k(j,r)}\theta = b_{m(j,r)}$. As in the proof of claim 1, consider the non-zero idempotent $x = a_{k(i,j)}a_{k'(j,r)}a^{-1}_{k(j,r)}a^{-1}_{k'(i,j)}$ of $S(\mathcal{T}_1)$. By claim 1, $a_{k'(j,r)}\theta = b_{l(j,r)}$ or $a_{k'(j,r)}\theta = b^{-1}_{l(r,j)}$ with $l \neq m$. If $a_{k'(j,r)}\theta = b_{l(j,r)}$, then $a_{k(j,r)}\theta \neq b_{l(j,r)}$ since θ is injective, and so $x\theta$ is an idempotent if and only if $a_{k(j,r)}\theta = b_{m(j,r)}$, because $a_{k'(i,j)}\theta$ only has two tiles. Similarly, if $a_{k'(j,r)}\theta = b^{-1}_{l(r,j)}$, then the injectivity of θ implies that $a_{k(j,r)}\theta \neq b^{-1}_{l(r,j)}$, so that $x\theta$ is an idempotent if and only if $a_{k(j,r)}\theta = b_{m(j,r)}$. Applying this argument now to $a_{k(j,r)}$ and $a_{k(r,s)}$, we conclude that $a_{k(r,s)}\theta = b_{m(r,s)}$.

Claim 3. Let $k' \neq k$. If $a_{k(i,j)}\theta = b_{m(i,j)}$, then the fact that $a_{k'(r,s)}\theta \neq b_{m(r,s)}$ and $a_{k'(r,s)}\theta \neq b^{-1}_{m(s,r)}$ is a consequence of Claim 1 since, by Claim 2, $a_{k(t,r)}\theta = b_{m(t,r)}$. Similarly, if $a_{k(i,j)}\theta = b^{-1}_{m(j,i)}$, then $a_{k'(r,s)}\theta \neq b_{m(r,s)}$ and $a_{k'(r,s)}\theta \neq b^{-1}_{m(s,r)}$ by Claims 1 and 2.

This proves statements (i) and (ii). □

The following lemma establishes a useful technical property of connected subsets in a hypercubic tiling. Recall from Chapter 3, Section 3.4, that we say that two vertices x and y of $\mathbb{Z}^n$ are adjacent if $x - y$ is either a standard basis vector or the negative of a standard basis vector of $\mathbb{Z}^n$.

Lemma 7.2.5. *Let A be a connected subset of $\mathbb{Z}^n$ with at least three elements and $a, b \in A$. Then there exist B_1, B_2 and B_3 connected proper subsets of A such that: $A = B_1 \cup B_2 \cup B_3$, $a \in B_1$, $b \in B_3$, $B_1 \cap B_2 \neq \emptyset$, and $B_2 \cap B_3 \neq \emptyset$.*

Proof. We discuss the possible cases. If $A \setminus \{a, b\}$ is connected and $a = b$, we take $B_1 = B_3 = \{a, a'\}$, where a' is a vertex adjacent to a, and $B_2 = A \setminus \{a\}$; if $A \setminus \{a, b\}$ is connected and $a \neq b$, then (i) if b is the only vertex adjacent to a, we can take $B_1 = B_3 = \{a, b\}$ and $B_2 = A \setminus \{a\}$; (ii) analogously, if a is the only vertex adjacent to b, we take $B_1 = B_3 = \{a, b\}$ and $B_2 = A \setminus \{b\}$; (iii) if there exist a vertex a' adjacent to a and a vertex b' adjacent to b, we can take $B_1 = \{a, a'\}$, $B_2 = A \setminus \{a, b\}$, and $B_3 = \{b, b'\}$. If $A \setminus \{a, b\}$ is not connected and $a = b$ or a and b are adjacent, we take $B_1 = C \cup \{a, b\}$, $B_2 = \{a, b\}$, and $B_3 = A \setminus C$, where C is a connected component of $A \setminus \{a, b\}$ (note that there exist at least two); if $A \setminus \{a, b\}$ is not connected and a and b are not adjacent, then we take $B_1 = C_a \cup \{a\}$, $B_2 = C \cup \{a, b\}$, and $B_3 = C_b \cup \{b\}$, where C_a (respectively, C_b) is the union of the connected components of $A \setminus \{a, b\}$ which contain a vertex adjacent to a (respectively, to b) and C a connected component of $A \setminus \{a, b\}$ containing a vertex adjacent to a and a vertex adjacent to b. □

Proof of necessity in Theorem 7.2.1. Suppose that $\mathcal{T}_1 = (\mathbb{Z}^n, \tau_1)$ and $\mathcal{T}_2 = (\mathbb{Z}^n, \tau_2)$ contain all finite, connected coloured subsets with four tiles and that $\theta\colon S(\mathcal{T}_1) \to S(\mathcal{T}_2)$ is an isomorphism. The existence of a bijection $\varphi\colon \Sigma_1 \to \Sigma_2$ has already been established. Consider the $n \times n$ matrix $M = [M_{km}]_{k,m \in \{1,\dots,n\}}$ defined by:

$$M_{km} = \begin{cases} 1, & \text{if } a_{k(i,j)}\theta = b_{m_{(\varphi(i),\varphi(j))}} \text{ for some } i, j \in \Sigma_1 \\ -1, & \text{if } a_{k(i,j)}\theta = b^{-1}_{m_{(\varphi(j),\varphi(i))}} \text{ for some } i, j \in \Sigma_1 \\ 0, & \text{otherwise}. \end{cases}$$

For $n = 1$, we have $M = [1]$ if $a_{(i,j)}\theta = b_{(\varphi(i),\varphi(j))}$ for some $i, j \in \Sigma_1$ (and hence for all $r, s \in \Sigma_1$, by Lemma 7.2.3) and $M = [-1]$ if $a_{(i,j)}\theta = b^{-1}_{(\varphi(j),\varphi(i))}$ for some $i, j \in \Sigma_1$ (and hence for all $r, s \in \Sigma_1$, again by Lemma 7.2.3). So M is 1×1 signed permutation matrix, corresponding, in the first case, to the identity automorphism and, in the second, to the reflection about the origin.

Now let $n \geqslant 2$. We claim that M has exactly one non-zero entry in each row and column (which is, by definition, either 1 or -1). In fact, if there were two distinct non-zero entries in the m^{th} column of the matrix M, say M_{km} and $M_{k'm}$, then $a_{k(i,j)}\theta = b_{m(\varphi(i),\varphi(j))}$ or $a_{k(i,j)}\theta = b^{-1}_{m(\varphi(j),\varphi(i))}$, for some $i, j \in \Sigma_1$, and $a_{k'(r,s)}\theta = b_{m(\varphi(r),\varphi(s))}$ or $a_{k'(r,s)}\theta = b^{-1}_{m(\varphi(s),\varphi(r))}$, for some $r, s \in \Sigma_1$, with $k \neq k'$, a contradiction by Lemma 7.2.4 (i). On the other hand, if there were two distinct non-zero entries in the k^{th} row of the matrix M, say M_{km} and $M_{km'}$, then $a_{k(i,j)}\theta = b_{m(\varphi(i),\varphi(j))}$ or $a_{k(i,j)}\theta = b^{-1}_{m(\varphi(j),\varphi(i))}$, for some $i, j \in \Sigma_1$, and $a_{k(r,s)}\theta = b_{m'(\varphi(r),\varphi(s))}$ or $a_{k(r,s)}\theta = b^{-1}_{m'(\varphi(s),\varphi(r))}$, for some $r, s \in \Sigma_1$, with $m \neq m'$, a contradiction by Lemma 7.2.4 (ii). So M is a signed permutation matrix and so it defines a symmetry σ of the n-dimensional hypercube.

Clearly, $0\theta = 0$; we show that, for each non-zero element $[a, A, b]$ in $S(\mathcal{T}_1)$, we have $[a, A, b]\theta = [a\varphi\sigma + v, A\varphi^*\sigma + v, b\varphi\sigma + v]$, for some translation $v \in \mathbb{Z}^n$, by induction on the number t of vertices of A. This is trivially true for $t = 1$ from the way φ was defined; for $t = 2$, it is true from the way φ was defined and from the way σ was chosen. In fact, if A has two vertices, then $[a, A, b]$ is one of the elements $a_{k(i,j)}$, $a^{-1}_{k(i,j)}$, $a_{k(i,j)}a^{-1}_{k(i,j)}$, or $a^{-1}_{k(i,j)}a_{k(i,j)}$, for some generator $a_{k(i,j)}$ of $S(\mathcal{T}_1)$. By Lemma 7.2.2, we know that

- if $[a, A, b] = a_{k(i,j)}$, then $[a, A, b]\theta = b_{m(\varphi(i),\varphi(j))}$ or $[a, A, b]\theta = b^{-1}_{m(\varphi(j),\varphi(i))}$;
- if $[a, A, b] = a^{-1}_{k(i,j)}$, then $[a, A, b]\theta = b^{-1}_{m(\varphi(i),\varphi(j))}$ or $[a, A, b]\theta = b_{m(\varphi(j),\varphi(i))}$;
- if $[a, A, b] = a_{k(i,j)}a^{-1}_{k(i,j)}$, then $[a, A, b]\theta = b_{m(\varphi(i),\varphi(j))}b^{-1}_{m(\varphi(i),\varphi(j))}$ or $[a, A, b]\theta = b^{-1}_{m(\varphi(j),\varphi(i))}b_{m(\varphi(j),\varphi(i))}$; and
- if $[a, A, b] = a^{-1}_{k(i,j)}a_{k(i,j)}$, then $[a, A, b]\theta = b^{-1}_{m(\varphi(i),\varphi(j))}b_{m(\varphi(i),\varphi(j))}$ or $[a, A, b]\theta = b_{m(\varphi(j),\varphi(i))}b^{-1}_{m(\varphi(j),\varphi(i))}$,

for some $m \in [n]$. In any case, the underlying finite, connected coloured subset of $[a, A, b]\theta$ is the Σ_2-coloured subset A of $\mathbb{Z}^n$ acted upon by the chosen symmetry σ with colours given by φ and in-tile (respectively, out-tile) defined in the same way from the in-tile (respectively, out-tile) of $[a, A, b]$. Therefore $[a, A, b]\theta = [a\varphi\sigma + v, A\varphi^*\sigma + v, b\varphi\sigma + v]$ for some translation $v \in \mathbb{Z}^n$. Now assume it is true for all finite, connected coloured subsets with $t' < t$ vertices and let A have $t > 2$ vertices. Then, by Lemma 7.2.5, there exist B_1, B_2 and B_3 finite, connected coloured subsets of $(\mathbb{Z}^n, \tau_1)$ (not necessarily distinct) with at most $t - 1$ vertices such that: $A = B_1 \cup B_2 \cup B_3$, $a \in B_1$, $b \in B_3$, $B_1 \cap B_2 \neq \emptyset$, and $B_2 \cap B_3 \neq \emptyset$. Let c be a vertex in $B_1 \cap B_2$ and d a vertex in $B_2 \cap B_3$. Then $[a, B_1, c][c, B_2, d][d, B_3, b] = [a, A, b]$. By assumption, there exist translations u, v and w such that $[a, B_1, c]\theta = [a\varphi\sigma + u, B_1\varphi^*\sigma + u, c\varphi\sigma + u]$, $[c, B_2, d]\theta = [c\varphi\sigma + v, B_2\varphi^*\sigma + v, d\varphi\sigma + v]$, and $[d, B_3, b]\theta = [d\varphi\sigma + w, B_3\varphi^*\sigma + w, b\varphi\sigma + w]$.

Thus, $[a, A, b]\theta = [a, B_1, c]\theta\,[c, B_2, d]\theta\,[d, B_3, b]\theta$ is a non-zero element, with

$$\begin{aligned}
[a, A, b]\theta &= [a\varphi\sigma + u, B_1\varphi^*\sigma + u, c\varphi\sigma + u][c\varphi\sigma + v, B_2\varphi^*\sigma + v, d\varphi\sigma + v] \\
&\qquad\qquad [d\varphi\sigma + w, B_3\varphi^*\sigma + w, b\varphi\sigma + w] \\
&= [a\varphi\sigma + u + u', (B_1\varphi^*\sigma + u + u') \cup (B_2\varphi^*\sigma + v + v'), d\varphi\sigma + v + v'] \\
&\qquad\qquad [d\varphi\sigma + w, B_3\varphi^*\sigma + w, b\varphi\sigma + w] \\
&= [a\varphi\sigma + u + u' + u'', \\
&\qquad\qquad \big(((B_1\varphi^*\sigma + u + u') \cup (B_2\varphi^*\sigma + v + v')) + u''\big) \cup (B_3\varphi^*\sigma + w + w''), \\
&\qquad\qquad\quad b\varphi\sigma + w + w''] \,,
\end{aligned}$$

for some translations u', v', u'', and w'' with

$$\begin{aligned}
c\varphi\sigma + u + u' &= c\varphi\sigma + v + v' \qquad \text{and} \\
d\varphi\sigma + v + v' + u'' &= d\varphi\sigma + w + w'' \,.
\end{aligned}$$

But then $u + u' = v + v'$ and $v + v' + u'' = w + w''$. Therefore, taking

$$z = u + u' + u'' = v + v' + u'' = w + w'' \,,$$

we get

$$[a, A, b]\theta = [a\varphi\sigma + z, (B_1 \cup B_2 \cup B_3)\varphi^*\sigma + z, b\varphi\sigma + z] = [a\varphi\sigma + z, A\varphi^*\sigma + z, b\varphi\sigma + z] \,,$$

with z a translation in $\mathbb{Z}^n$ such that $A\varphi^*\sigma + z$ is a finite, connected Σ_2-coloured subset of $(\mathbb{Z}^n, \tau_2)$, as claimed. □

We now turn to the proof of sufficiency in Theorem 7.2.1. The following simple lemma is folklore; we include a proof for completeness.

Lemma 7.2.6. *Let S and T be inverse semigroups with zero and $\theta\colon S \to T$ and $\eta\colon T \to S$ mutually inverse 0-restricted pre-homomorphisms. Then θ and η are isomorphisms.*

Proof. The fact that θ and η are mutually inverse implies that they are bijections. Since, by assumption, $(s_1s_2)\theta = s_1\theta s_2\theta$ for all $s_1, s_2 \in S$ such that $s_1s_2 \neq 0$, to show that θ is a homomorphism, we need only prove that $(s_1s_2)\theta = s_1\theta s_2\theta$ when $s_1s_2 = 0$ as well. In fact, if $s_1\theta s_2\theta \neq 0$, then, because η is 0-restricted and $\theta\eta$ is the identity map on S, we have

$$0 \neq (s_1\theta s_2\theta)\eta = s_1\theta\eta\, s_2\theta\eta = s_1s_2 = 0 \,,$$

a contradiction. Therefore θ is an isomorphism and, similarly, so is η. □

Proof of sufficiency in Theorem 7.2.1. Suppose there exists a bijection $\varphi\colon \Sigma_1 \to \Sigma_2$ and a symmetry σ of G_n such that if A is a finite, connected Σ_1-coloured subset of $(\mathbb{Z}^n, \tau_1)$ then $A\varphi^*\sigma + v$ is a finite, connected Σ_2-coloured subset of $(\mathbb{Z}^n, \tau_2)$ for some $v \in \mathbb{Z}^n$ and that if B is a finite, connected Σ_2-coloured subset of $(\mathbb{Z}^n, \tau_2)$ then $B(\varphi^{-1})^*\sigma^{-1} + v$ is a finite, connected Σ_1-coloured subset of $(\mathbb{Z}^n, \tau_1)$ for some $v \in \mathbb{Z}^n$. Consider the map $\theta\colon S(\mathcal{T}_1) \to S(\mathcal{T}_2)$ defined on the non-zero elements of $S(\mathcal{T}_1)$ by $[a, A, b]\theta = [a\varphi\sigma + v, A\varphi^*\sigma + v, b\varphi\sigma + v]$, where v is a translation of $\mathbb{Z}^n$ such that $A\varphi^*\sigma + v$ is a finite, connected Σ_2-coloured subset of $(\mathbb{Z}^n, \tau_2)$.

We begin by proving that θ is well-defined. Let $[a, A, b]$ be a non-zero element of $S(\mathcal{T}_1)$. Since by assumption there is a translation v for which $A\varphi^*\sigma + v$ is a finite, connected Σ_2-coloured subset of $(\mathbb{Z}^n, \tau_2)$ and since $a\varphi\sigma + v$ and $b\varphi\sigma + v$ are vertices of this subset, then $[a\varphi\sigma + v, A\varphi^*\sigma + v, a\varphi\sigma + v]$ is a non-zero element of $S(\mathcal{T}_2)$. If w is another translation of $\mathbb{Z}^n$ such that $A\varphi^*\sigma + w$ is a finite, connected Σ_2-coloured subset of $(\mathbb{Z}^n, \tau_2)$, then $v - w \in \mathbb{Z}^n$ and $A\varphi^*\sigma + v = A\varphi^*\sigma + w + (v - w)$, so that $[a\varphi\sigma + v, A\varphi^*\sigma + v, b\varphi\sigma + v] = [a\varphi\sigma + w, A\varphi^*\sigma + w, b\varphi\sigma + w]$, by definition of the elements of a tiling semigroup. Therefore, the definition of θ does not depend on the choice of the translation v. Finally, if $[a, A, b]$ and $[c, C, d]$ represent the same element of $S(\mathcal{T}_1)$, then there exists a translation u of $\mathbb{Z}^n$ such that $A = C + u$, $a = c + u$ and $b = d + u$. Let v and w be such that

$$\begin{aligned}
[a, A, b]\theta &= [a\varphi\sigma + v, A\varphi^*\sigma + v, b\varphi\sigma + v] \quad \text{and} \\
[c, C, d]\theta &= [c\varphi\sigma + w, C\varphi^*\sigma + w, d\varphi\sigma + w]\,.
\end{aligned}$$

Then, since σ is a linear transformation of $\mathbb{Z}^n$, we have that $(C + u)\varphi^*\sigma = C\varphi^*\sigma + u\sigma$, with $u\sigma \in \mathbb{Z}^n$, and so

$$\begin{aligned}
[a, A, b]\theta &= [(c + u)\varphi\sigma + v, (C + u)\varphi^*\sigma + v, (d + u)\varphi\sigma + v] \\
&= [c\varphi\sigma + u\sigma + v, C\varphi^*\sigma + u\sigma + v, d\varphi\sigma + u\sigma + v] \\
&= [c\varphi\sigma + w + (u\sigma + v - w), C\varphi^*\sigma + w + (u\sigma + v - w), d\varphi\sigma + w + (u\sigma + v - w)] \\
&= [c\varphi\sigma + w, C\varphi^*\sigma + w, d\varphi\sigma + w] \\
&= [c, C, d]\theta\,.
\end{aligned}$$

So θ is well-defined.

By definition, θ is 0-restricted. Next, we show that θ is a pre-homomorphism. Let $[a, A, b]$ and $[c, C, d]$ be non-zero elements of $S(\mathcal{T}_1)$ such that $[a, A, b][c, C, d] \neq 0$. Then $[a, A, b][c, C, d] = [a + u, (A + u) \cup (C + u'), d + u']$, for some translations u and u' of $\mathbb{Z}^n$ such that $(A + u) \cup (C + u')$ is a finite, connected Σ_1-coloured subset of $(\mathbb{Z}^n, \tau_1)$ and $b + u = c + u'$. Let $v, v', w \in \mathbb{Z}^n$ be such that

$$\begin{aligned}
[a, A, b]\theta &= [a\varphi\sigma + v, A\varphi^*\sigma + v, b\varphi\sigma + v]\,, \\
[c, C, d]\theta &= [c\varphi\sigma + v', C\varphi^*\sigma + v', d\varphi\sigma + v']
\end{aligned}$$

and

$$\begin{aligned}
([a, A, b]\,[c, C, d])\theta &= [a + u, (A + u) \cup (C + u'), d + u']\theta = \\
&= [(a + u)\varphi\sigma + w, \big((A + u) \cup (C + u')\big)\varphi^*\sigma + w, (d + u')\varphi\sigma + w]\,.
\end{aligned}$$

Let $z = u\sigma + w - v$ and $z' = u'\sigma + w - v'$, so that $z, z' \in \mathbb{Z}^n$ and $u\sigma + w = v + z$ and $u'\sigma + w = v' + z'$. Then

$$\begin{aligned}\big((A+u)\cup(C+u')\big)\varphi^*\sigma + w &= \big((A+u)\varphi^*\sigma + w\big)\cup\big((C+u')\varphi^*\sigma + w\big)\\ &= \big(A\varphi^*\sigma + u\sigma + w\big)\cup\big(C\varphi^*\sigma + u'\sigma + w\big)\\ &= \big(A\varphi^*\sigma + v + z\big)\cup\big(C\varphi^*\sigma + v' + z'\big)\end{aligned}$$

is a finite, connected Σ_1-coloured subset of $(\mathbb{Z}^n, \tau_2)$. As $b + u = c + u'$, we also have

$$\begin{aligned}b\varphi\sigma + v + z &= b\varphi\sigma + u\sigma + w\\ &= (b+u)\varphi\sigma + w\\ &= (c+u')\varphi\sigma + w\\ &= c\varphi\sigma + u'\sigma + w\\ &= c\varphi\sigma + v' + z'\,.\end{aligned}$$

Therefore $[a\varphi\sigma + v, A\varphi^*\sigma + v, b\varphi\sigma + v][c\varphi\sigma + v', C\varphi^*\sigma + v', d\varphi\sigma + v'] \neq 0$ and

$$\begin{aligned}[a,A,b]\theta\,[c,C,d]\theta &= [a\varphi\sigma + v, A\varphi^*\sigma + v, b\varphi\sigma + v][c\varphi\sigma + v', C\varphi^*\sigma + v', d\varphi\sigma + v']\\ &= [a\varphi\sigma + v + z, \big(A\varphi^*\sigma + v + z\big)\cup\big(B\varphi^*\sigma + v' + z'\big), d\varphi\sigma + v' + z']\\ &= [a\varphi\sigma + u\sigma + w, \big(A\varphi^*\sigma + u\sigma + w\big)\cup\big(B\varphi^*\sigma + u'\sigma + w\big), d\varphi\sigma + u'\sigma + w]\\ &= [(a+u)\varphi\sigma + w, \big((A+u)\cup(C+u')\big)\varphi^*\sigma + w, (d+u')\varphi\sigma + w]\\ &= ([a,A,b]\,[c,C,d])\,\theta\,.\end{aligned}$$

Hence θ is a pre-homomorphism.

Dually, consider the map $\eta\colon S(\mathcal{T}_2) \to S(\mathcal{T}_1)$ defined by $0\eta = 0$ and, for all non-zero elements $[a, A, b]$ in $S(\mathcal{T}_2)$, by $[a, A, b]\eta = [a\varphi^{-1}\sigma^{-1} + v, A(\varphi^{-1})^*\sigma^{-1} + v, b\varphi^{-1}\sigma^{-1} + v]$, where v is a translation of $\mathbb{Z}^n$ such that $A(\varphi^{-1})^*\sigma^{-1}+v$ is a finite, connected Σ_1-coloured subset of $(\mathbb{Z}^n, \tau_1)$. Then η is a well-defined 0-restricted pre-homomorphism.

We claim that $\theta\eta = id_{S(\mathcal{T}_1)}$ and that $\eta\theta = id_{S(\mathcal{T}_2)}$. Since θ and η are 0-restricted by definition, then $0(\theta\eta) = 0$ and $0(\eta\theta) = 0$. Now let $[a, A, b]$ be a non-zero element from $S(\mathcal{T}_1)$. Then $[a, A, b]\theta = [a\varphi\sigma + v, A\varphi^*\sigma + v, b\varphi\sigma + v]$, with $A\varphi^*\sigma + v$ a finite, connected Σ_2-coloured subset of $(\mathbb{Z}^n, \tau_2)$, and

$$[a,A,b]\theta\eta = [(a\varphi\sigma + v)\varphi^{-1}\sigma^{-1} + w, (A\varphi^*\sigma + v)\,(\varphi^{-1})^*\sigma^{-1} + w, (b\varphi\sigma + v)\varphi^{-1}\sigma^{-1} + w]\,,$$

with $(A\varphi^*\sigma + v)\,(\varphi^{-1})^*\sigma^{-1} + w$ a finite, connected Σ_1-coloured subset of $(\mathbb{Z}^n, \tau_1)$. As the changing of colours (by $(\varphi^{-1})^*$) and the performance of a symmetry (by σ) commute, we have

$$\varphi^*\sigma(\varphi^{-1})^*\sigma^{-1} = \varphi^*(\varphi^{-1})^*\sigma\sigma^{-1} = (\varphi\varphi^{-1})^* id = id\,,$$

and so

$$(A\varphi^*\sigma + v)\,(\varphi^{-1})^*\sigma^{-1} + w = A\varphi^*\sigma(\varphi^{-1})^*\sigma^{-1} + v\sigma^{-1} + w = A + v\sigma^{-1} + w\,,$$

with $v\sigma^{-1} + w \in \mathbb{Z}^n$. Thus,

$$\begin{aligned}
[a, A, b]\theta\eta &= [(a\varphi\sigma + v)\varphi^{-1}\sigma^{-1} + w, (A\varphi^*\sigma + v)\,(\varphi^{-1})^*\sigma^{-1} + w, (b\varphi\sigma + v)\varphi^{-1}\sigma^{-1} + w] \\
&= [a + v\sigma^{-1} + w, A + v\sigma^{-1} + w, b + v\sigma^{-1} + w] \\
&= [a, A, b]\,.
\end{aligned}$$

Therefore $\theta\eta = id_{S(\mathcal{T}_1)}$ and, similarly, $\eta\theta = id_{S(\mathcal{T}_2)}$. Hence, θ and η are mutually inverse.

By Lemma 7.2.6, we conclude that θ is an isomorphism. □

The next result rephrases Theorem 7.2.1 in terms of the tiling languages of hypercubic tilings. In the next subsection, this will be of use to obtain a neat characterization for the one-dimensional case.

Recall that we denote by A_0 the minimum element of a finite subset A of $\mathbb{Z}^n$ under the order $\leqslant$ from Lemma 4.1.1.

Corollary 7.2.7. *Let $\mathcal{T}_1 = (\mathbb{Z}^n, \tau_1)$ and $\mathcal{T}_2 = (\mathbb{Z}^n, \tau_2)$ be n-dimensional hypercubic tilings with languages L_1 and L_2, respectively. Suppose that the mapping $\psi\colon L_1 \to L_2$ defined by $P\psi = P\varphi^*\sigma - (P\sigma)_0$ is a bijection, for some bijection $\varphi\colon \Sigma_1 \to \Sigma_2$ and symmetry σ from G_n. Then $\zeta\colon S(L_1) \to S(L_2)$, defined on the non-zero elements by*

$$(p, P, q)\zeta = (p\varphi\sigma - (P\sigma)_0, P\psi, q\varphi\sigma - (P\sigma)_0)\,,$$

is an isomorphism. Conversely, if both L_1 and L_2 contain all words up to four letters, then every isomorphism of $S(L_1)$ onto $S(L_2)$ is of this form for some bijection $\varphi\colon \Sigma_1 \to \Sigma_2$ and some symmetry σ of G_n.

Proof. First, note that the fact that each $\sigma \in G_n$ defines a linear mapping and that the order $\leqslant$ of $\mathbb{Z}^n$ is compatible with addition implies that, for all $P \in L_1$ and $u \in \mathbb{Z}^n$ such that $P + u$ is a (finite and connected) Σ_1-coloured subset of $(\mathbb{Z}^n, \tau_1)$, we have

$$\begin{aligned}
(P + u)\varphi^*\sigma - ((P + u)\varphi^*\sigma)_0 &= P\varphi^*\sigma + u\sigma - (P\varphi^*\sigma + u\sigma)_0 \\
&= P\varphi^*\sigma + u\sigma - (P\varphi^*\sigma)_0 - u\sigma \\
&= P\varphi^*\sigma - (P\sigma)_0\,.
\end{aligned}$$

So suppose that $\varphi\colon \Sigma_1 \to \Sigma_2$ is a bijection and σ a symmetry from G_n such that the map $\psi\colon L_1 \to L_2$ defined by $P\psi = P\varphi^*\sigma - (P\sigma)_0$ is a bijection. We claim that $S(\mathcal{T}_1)$ and $S(\mathcal{T}_2)$ are isomorphic. Let A be a finite, connected Σ_1-coloured subset of $(\mathbb{Z}^n, \tau_1)$. Then $A - A_0 \in L_1$, and so $(A - A_0)\psi = A\varphi^*\sigma - (A\sigma)_0 \in L_2$. Thus, $A\varphi^*\sigma - (A\sigma)_0 = B - B_0$ for some finite, connected Σ_2-coloured subset B of $(\mathbb{Z}^n, \tau_2)$. Therefore, $A\varphi^*\sigma + v$, with $v = B_0 - (A\sigma)_0 \in \mathbb{Z}^n$, is a finite, connected Σ_2-coloured subset of $(\mathbb{Z}^n, \tau_2)$. From the definition of the bijection ψ, it is easy to check that $\psi^{-1}\colon L_2 \to L_1$ is defined by $Q\psi^{-1} = Q(\varphi^{-1})^*\sigma^{-1} - (Q\sigma^{-1})_0$ for all $Q \in L_2$; now using ψ^{-1}, we can show just as before that, for each finite, connected Σ_2-coloured subset B of $(\mathbb{Z}^n, \tau_2)$ there exists $v \in \mathbb{Z}^n$ such that $B(\varphi^{-1})^*\sigma^{-1} + v$ is a finite, connected Σ_1-coloured subset of $(\mathbb{Z}^n, \tau_1)$. Hence, by Theorem 7.2.1, $S(\mathcal{T}_1) \simeq S(\mathcal{T}_2)$ via the isomorphism $\theta\colon S(\mathcal{T}_1) \to S(\mathcal{T}_2)$ defined on the non-zero elements by $[a, A, b]\theta = [a\varphi\sigma + v, A\varphi^*\sigma + v, b\varphi\sigma]$ (with $v \in \mathbb{Z}^n$). But then, by Theorem 4.2.3, the composition $\zeta = \phi_1^{-1}\theta\phi_2\colon S(L_1) \to S(L_2)$ (where $\phi_1\colon S(\mathcal{T}_1) \to S(L_1)$ and $\phi_2\colon S(\mathcal{T}_2) \to S(L_2)$ are the maps in Theorem 4.2.3) is an isomorphism as well and it is not hard to check that, for each non-zero $(p, P, q) \in S(L_1)$, we have

$$(p, P, q)\zeta = (p\varphi\sigma - (P\sigma)_0, P\varphi^*\sigma - (P\sigma)_0, q\varphi\sigma - (P\sigma)_0)\,,$$

that is, $(p, P, q)\zeta = (p\varphi\sigma - (P\sigma)_0, P\psi, q\varphi\sigma - (P\sigma)_0)$, as claimed.

$$\begin{tikzcd} S(\mathcal{T}_1) \arrow[r, "\theta"] & S(\mathcal{T}_2) \arrow[d, "\phi_2"] \\ S(L_1) \arrow[u, "{\phi_1}^{-1}"] \arrow[r, dashed, "\zeta"'] & S(L_2) \end{tikzcd}$$

Conversely, assume that L_1 and L_2 contain all words up to four letters and that we have an isomorphism $\zeta\colon S(L_1) \to S(L_2)$. Then, by Theorem 4.2.3, we have that the map $\theta = \phi_1\zeta\phi_2^{-1}\colon S(\mathcal{T}_1) \to S(\mathcal{T}_2)$ is an isomorphism. By Theorem 7.2.1, there must exist a bijection $\varphi\colon \Sigma_1 \to \Sigma_2$ and a symmetry $\sigma \in G_n$ such that, for all non-zero $[a, A, b] \in S(\mathcal{T}_1)$,

$$[a, A, b]\theta = [a\varphi\sigma + v, A\varphi^*\sigma + v, b\varphi\sigma + v]\,,$$

or, equivalently,

$$[a, A, b]\phi_1\zeta = [a\varphi\sigma + v, A\varphi^*\sigma + v, b\varphi\sigma + v]\phi_2\,.$$

But then

$$(a - A_0, A - A_0, b - A_0)\zeta = (a\varphi\sigma - (A\sigma)_0, A\varphi^*\sigma - (A\sigma)_0, b\varphi\sigma - (A\sigma)_0)\,,$$

which gives us the desired form for ζ. In particular, the map $\psi\colon L_1 \to L_2$ defined by $P\psi = P\varphi^*\sigma - (P\sigma)_0$ is a bijection. $\square$

Although we shall not give a proof for it, we note that the previous corollary can be generalized for arbitrary n-dimensional factorial languages:

Theorem 7.2.8. *Let L_1 and L_2 be n-dimensional factorial languages over Σ_1 and Σ_2, respectively. Suppose that the mapping $\psi\colon L_1 \to L_2$ defined by $P\psi = P\varphi^*\sigma - (P\sigma)_0$ is a bijection, for some bijection $\varphi\colon \Sigma_1 \to \Sigma_2$ and symmetry σ from G_n. Then $\zeta\colon S(L_1) \to S(L_2)$, defined on the non-zero elements by*

$$(p, P, q)\zeta = (p\varphi\sigma - (P\sigma)_0, P\psi, q\varphi\sigma - (P\sigma)_0)\,,$$

is an isomorphism. Conversely, if both L_1 and L_2 contain all words up to four letters, then every isomorphism of $S(L_1)$ onto $S(L_2)$ is of this form for some bijection $\varphi\colon \Sigma_1 \to \Sigma_2$ and some symmetry σ of G_n.

Corollary 7.2.7 and Theorem 7.2.8 enable us to characterize the group of automorphisms of an n-dimensional hypercubic tiling semigroup and the group of automorphisms of the semigroup $S(L)$ associated with an arbitrary n-dimensional factorial language L, respectively. In fact, let L be either the language of an n-dimensional hypercubic tiling or an arbitrary n-dimensional factorial language, over Σ, containing all words up to four letters. From what we have just seen, $\zeta\colon S(L) \to S(L)$ is an isomorphism if and only if the map $\psi_{(\varphi,\sigma)}\colon L \to L$ defined by $P\psi_{(\varphi,\sigma)} = P\varphi^*\sigma - (P\sigma)_0$ is a bijection for some bijection $\varphi\colon \Sigma \to \Sigma$ and some symmetry σ from G_n. Each automorphism of $S(L)$ is thus associated with a pair (φ, σ), where φ is a permutation of Σ, that is, $\varphi \in S_{|\Sigma|}$, and $\sigma \in G_n$. Moreover, if ζ and η are automorphisms of $S(L)$ with associated bijections $\psi_{(\varphi,\sigma)}$ and $\psi_{(\phi,\omega)}$, respectively, for some $\varphi, \phi \in S_{|\Sigma|}$ and $\sigma, \omega \in G_n$, then the automorphism $\zeta\eta$ has associated bijection $\psi_{(\varphi\phi,\sigma\omega)}$. To see this, let $P \in L$. Then

$$\begin{aligned}
P\psi_{(\varphi,\sigma)}\psi_{(\phi,\omega)} &= (P\varphi^*\sigma - (P\sigma)_0)\psi_{(\phi,\omega)} \\
&= (P\varphi^*\sigma - (P\sigma)_0)\phi^*\omega - ((P\varphi^*\sigma - (P\sigma)_0)\omega)_0 \\
&= P\varphi^*\sigma\phi^*\omega - (P\sigma)_0\phi^*\omega - (P\varphi^*\sigma\omega - (P\sigma)_0\omega)_0 \\
&= P\varphi^*\sigma\phi^*\omega - (P\sigma)_0\omega - (P\sigma\omega)_0 + (P\sigma)_0\omega \\
&= P\varphi^*\sigma\phi^*\omega - (P\sigma\omega)_0 \,,
\end{aligned}$$

from the linearity of ω. Now, $P\varphi^* = (P,\alpha)\varphi^* = (P, \varphi\alpha)$, where $\varphi\alpha\colon P \to \Sigma$ is defined by $(\varphi\alpha)(x) = \varphi(\alpha(x))$, for all $x \in P$, and, likewise, $P\varphi^*\phi^* = (P,\alpha)\varphi^*\phi^* = (P, \phi\varphi\alpha)$, where $\phi\varphi\alpha\colon P \to \Sigma$ is defined by $(\phi\varphi\alpha)(x) = \phi(\varphi(\alpha(x))) = (\phi\varphi)(\alpha(x))$, for all $x \in P$. Thus, and since the permutation of colours and the performance of a symmetry commute, we have $P\varphi^*\sigma\phi^*\omega = P\varphi^*\phi^*\sigma\omega = P(\varphi\phi)^*\sigma\omega$. Therefore,

$$P\psi_{(\varphi,\sigma)}\psi_{(\phi,\omega)} = P(\varphi\phi)^*(\sigma\omega) - (P\sigma\omega)_0 = P\psi_{(\varphi\phi,\sigma\omega)} \,.$$

Hence,

Proposition 7.2.9. *Let L be an n-dimensional factorial language over an alphabet Σ containing all words up to four letters. Then $\operatorname{Aut}(S(L))$ is isomorphic to the subgroup of $S_{|\Sigma|} \times G_n$ consisting of all pairs (φ, σ) such that the map $\psi_{(\varphi,\sigma)}\colon L \to L$ defined by $P\psi_{(\varphi,\sigma)} = P\varphi^*\sigma - (P\sigma)_0$ is a bijection.*

Notice that, if $\psi_{(\varphi,\sigma)}$ defines a map from a language L to itself, then it necessarily defines an injective map. Thus, such a map defines a bijection if and only if $L\psi_{(\varphi,\sigma)} = L$.

To compute the image under ζ of a word from the language of a hypercubic tiling, one needs to determine the minimum element of the word acted upon by a symmetry from G_n. Next, we address the problem of finding the minimum element A_0 of a Σ-coloured subset (A, α) of $\mathbb{Z}^n$ acted upon by a symmetry from G_n. This is difficult to do in general. It becomes easier if we deal with a particular type of Σ-coloured subset.

Definition 7.2.10. Let r be a positive integer. We call *basic hypercube of length r of $\mathbb{Z}^n$* to the subset of $\mathbb{Z}^n$

$$B_r = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \colon 0 \leqslant x_i \leqslant r \text{ for all } i \in \{1, \ldots, n\}\},$$

where r is a non-negative integer.

In dimension two, the subset of a language consisting only of words of the form (B_r, α), with r be a positive integer, is a particular example of the two-dimensional languages considered in [17]. This class plays in fact an important role in that context, since many other languages can be derived from languages consisting of squares, by means of the two-dimensional analogues of concatenation (row and column concatenation), star operation (row and column closure), and the Boolean operators.

The next result solves the problem of finding the minimum element of a basic hypercube acted upon by a symmetry from G_n.

Proposition 7.2.11. *Let $\sigma \in G_n$ and let $M = [M_{ij}]_{i,j\in\{1,\ldots,n\}}$ be the matrix of σ with respect to the standard basis of $\mathbb{Z}^n$. Let $u = (u_1, \ldots, u_n) \in \mathbb{Z}^n$ be such that $u_i = 1$ if there exists j such that $M_{ij} = -1$ (that is, there exists an entry equal to -1 in the i^{th} row of M) and $u_i = 0$ otherwise. Then $(B_r\sigma)_0 = ruM = r(u\sigma)$, for each positive integer r.*

Proof. We shall prove that, for all $x \in B_1$, we have $(x - u)\sigma \geqslant (0, \ldots, 0)$, with respect to the order on $\mathbb{Z}^n$ considered in Lemma 4.1.1. Therefore, from the linearity of σ and the fact that $u \in B_1$, this implies that $u\sigma = (B_1\sigma)_0$ and hence that $r(u\sigma) = (B_r\sigma)_0$.

So let $x \in B_1$. Then $x = (x_1, \ldots, x_n)$ with $x_i = 0$ or $x_i = 1$, for all $i \in [n]$. Thus,

$$x_i - u_i = \begin{cases} 0 & \text{if } x_i = u_i \\ 1 & \text{if } x_i = 1 \text{ and } u_i = 0 \\ -1 & \text{if } x_i = 0 \text{ and } u_i = 1\,. \end{cases} \tag{7.1}$$

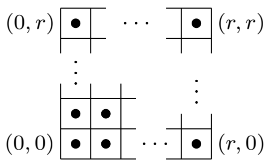
We will show that every component of $(x - u)\sigma$ is non-negative; this yields the desired conclusion. Now,

$$(x - u)\sigma = (x - u)M = \left(\sum_{i=1}^{n} M_{i1}\,(x_i - u_i), \ldots, \sum_{i=1}^{n} M_{in}\,(x_i - u_i) \right).$$

Fix $j \in [n]$. To prove that $\sum_{i=1}^{n} M_{ij}\,(x_i - u_i) \geqslant 0$ we actually show that $M_{ij}\,(x_i - u_i) \geqslant 0$ for all $i \in [n]$. So let $i \in [n]$ and, in order to obtain a contradiction, suppose that $M_{ij}\,(x_i - u_i) = -1$. Then, either (i) $M_{ij} = -1$ and $x_i - u_i = 1$ or (ii) $M_{ij} = 1$ and $x_i - u_i = -1$. In case (i), we have $u_i = 1$, by definition of u, and so $x_i - u_i \neq 1$ by (7.1), a contradiction; in case (ii), we have $u_i = 1$ by (7.1), which implies that $M_{ik} = -1$ for some k, by definition of u. But then $M_{ik} = -1$ and $M_{ij} = 1$, a contradiction as M is a signed permutation matrix. Therefore, $M_{ij}\,(x_i - u_i) \neq -1$, and so $M_{ij}\,(x_i - u_i)$ is either 0 or 1, for all $i, j \in [n]$. Hence, $((x - u)\sigma)_j = \sum_{i=1}^{n} M_{ij}\,(x_i - u_i) \geqslant 0$, for all $j \in [n]$. We conclude that $(x - u)\sigma \geqslant (0, \ldots, 0)$, as claimed. $\square$

We end this section with an example that illustrates the computation proved in the previous proposition.

Example 7.2.12. In the two-dimensional case, the basic hypercube of length r is the subset $B_r = \big\{(x, y) \in \mathbb{Z}^2 \colon 0 \leqslant x, y \leqslant r\big\}$ of $\mathbb{Z}^2$:



As we saw earlier in this section, the matrix that represents $\sigma_{y=x}$, the reflection about the straight line with equation $y = x$, with respect to the standard basis of $\mathbb{Z}^2$, is

$$M_{\sigma_{y=x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Thus, $u_{\sigma_{y=x}} = (0, 0)$, and so

$$(B_r\sigma)_0 = r u_{\sigma_{y=x}} M_{\sigma_{y=x}} = r \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = r \begin{pmatrix} 0 & 0 \end{pmatrix} = (0, 0) .$$

Now for $\sigma_{\frac{\pi}{2}}$, the rotation about the origin of $\frac{\pi}{2}$, since

$$M_{\sigma_{\frac{\pi}{2}}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} ,$$

then $u_{\sigma_{\frac{\pi}{2}}} = (1, 0)$, and so

$$(B_r\sigma)_0 = r M_{\sigma_{\frac{\pi}{2}}} u_{\sigma_{\frac{\pi}{2}}} = r \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = r \begin{pmatrix} 0 & -1 \end{pmatrix} = (0, -r) ,$$

as expected.

7.3 Isomorphic one-dimensional tiling semigroups

In this section, we specify to one-dimensional tiling semigroups and inverse semigroups associated with (one-dimensional) factorial languages, for which we will be able to derive an explicit expression for the isomorphism. In addition, we also show that the conditions in Theorem 7.2.1 can be somewhat loosened in the case of one-dimensional tiling semigroups, but not in the case of the inverse semigroups associated with (one-dimensional) factorial languages.

As the group G_1 has only two elements, the identity element and the reflection about the origin, in the one-dimensional case Corollary 7.2.7 states that the inverse semigroups $S(L_1)$ and $S(L_2)$ associated with the tilings languages $L_1 = L(\mathcal{T}_1)$ and $L_2 = L(\mathcal{T}_2)$ containing all words with four letters — that is, all connected coloured subsets (A, α) of $\mathbb{Z}$ with $A_0 = 0$

and $|A| = 4$ — are isomorphic if and only if there is a bijection $\varphi\colon \Sigma_1 \to \Sigma_2$ such that either $L_2 = L_1\varphi^*$ or $L_2 = L_1^{op}\varphi^*$, where L_1^{op} is the language of the reversals $w^{op} = a_l \dots a_1$ of the words $w = a_1 \dots a_l \in L_1$. Moreover, it gives the explicit form for the isomorphism $\zeta\colon S(L_1) \to S(L_2)$. In the first case, where σ is the identity and $L_2 = L_1\varphi^*$, we simply have $0\zeta = 0$ and

$$(i, w, j)\zeta = (i, w\varphi^*, j)\,,$$

as $(w\varphi^*\sigma)_0 = 0$, for all non-zero $(i, w, j) \in S(L_1)$. In the second, since $(w\varphi^*\sigma)_0 = -|w| + 1$ when σ is the reflection about the origin, then $0\zeta = 0$ and

$$(i, w, j)\zeta = (|w| - i + 1, (w\varphi^*)^{op}, |w| - j + 1)\,,$$

for all non-zero $(i, w, j) \in S(L_1)$. (Note that $(w\sigma)_0$ is the position of leftmost letter of $w\sigma$.) Analogous conclusions can be drawn for one-dimensional factorial languages from Theorem 7.2.8. And, ignoring the bijective correspondence between colours, this is exactly the isomorphism that showed that $S(L) \simeq S(L^{op})$ in Example 5.3.2 from Chapter 5. As we noticed there, now we have proved that, for any factorial languages L_1 and L_2 containing all words up to four letters, $\zeta\colon S(L_1) \to S(L_2)$ is an isomorphism if and only if ζ is one of the above.

In fact, we did not make use of the condition on the existence, in the tiling, of all patterns with four tiles — or all words up to four letters in the tiling language — in the one-dimensional case (cf. proof of Lemma 7.2.3). However, in the more general framework of inverse semigroups associated with factorial languages, the result does not necessarily hold without an additional condition, only slightly weaker that the one for the n-dimensional case: it is very easy to see that the proof of Lemma 7.2.3 works for factorial languages if we assume that these contain all words of length 3 over their alphabets. The next example shows that, without this restriction, it is possible for $S(L_1)$ and $S(L_2)$ to be isomorphic even though there is no bijection $\varphi\colon \Sigma_1 \to \Sigma_2$ such that either $L_2 = L_1\varphi^*$ or $L_2 = L_1^{op}\varphi^*$.

Example 7.3.1. Let

$$L = \{b, a^n, a^n b\colon n \in \mathbb{N}\}$$

and

$$K = \{b, c^n, c^n b\colon n \in \mathbb{N}\}\ .$$

Since every factor of $a^n b^i$, with $n \in \mathbb{N}$ and $i \in \{0, 1\}$, is a word of the form $a^m b^j$, with $m \in \mathbb{N}_0$ and $j \in \{0, 1\}$ — where, as usual, x^0 (with $x \in \Sigma$) denotes the empty word — we conclude that it belongs to L. Similarly, every factor of $b^i c^n$, with $n \in \mathbb{N}$ and $i \in \{0, 1\}$, also belongs to L. Thus, L is factorial and, analogously, so is K. Since $L \cap (K \cap K^{op}) = \{b\}$, we have that $L \cap (K \cup K^{op})$ contains no word of length two. Hence, from Example 5.3.4 in Chapter 5 we conclude that $S(L_1) \simeq S(L_2)$, where $L_1 = L \cup K$ and $L_2 = L \cup K^{op}$ are both languages over $\Sigma = \{a, b, c\}$:

$$L_1 = \{b, a^n, a^n b, c^n, bc^n : n \in \mathbb{N}\}$$

and

$$L_2 = \{b, a^n, a^n b, c^n, c^n b \colon n \in \mathbb{N}\} \ .$$

We will show that, nonetheless, there exists no bijection φ from Σ onto Σ such that $L_2 = L_1\varphi^*$ or $L_2 = L_1^{op}\varphi^*$. Recall from the definition of the extension φ^* of a map φ that $(a_1 a_2 \dots a_n)\varphi^* = a_1\varphi\, a_2\varphi \dots a_n\varphi$. Thus, φ^* preserves the length of a word. Now, the only two-letter words in L_1 using distinct letters are ab and bc and the only two-letter words in L_2 using distinct letters are ab and cb. Thus, if a bijection $\varphi\colon \Sigma \to \Sigma$ was such $L_2 = L_1\varphi^*$, we would have:

- if $(ab)\varphi^* = ab$, then $(bc)\varphi^* = cb$, and so $b = b\varphi = c$; and
- if $(ab)\varphi^* = cb$, then $(bc)\varphi^* = ab$, and so $b = b\varphi = a$,

a contradiction in both cases. Similarly, if a bijection $\varphi\colon \Sigma \to \Sigma$ was such $L_2 = L_1^{op}\varphi^*$, we would have:

- if $(ab)^{op}\varphi^* = (ba)\varphi^* = ab$, then $(bc)^{op}\varphi^* = (cb)\varphi = cb$, and so $a = b\varphi = b$; and
- if $(ab)^{op}\varphi^* = (ba)\varphi^* = cb$, then $(bc)^{op}\varphi^* = (cb)\varphi = ab$, and so $c = b\varphi = b$,

again a contradiction in both cases. Hence, we have $L_2 \neq L_1\varphi^*$ and $L_2 \neq L_1^{op}\varphi^*$ for all bijections $\varphi\colon \Sigma_1 \to \Sigma_2$.

Appendix: The full symmetry group of the n-dimensional hypercube

In Chapter 7, the full symmetry group G_n of an n-dimensional hypercube played an important role in our investigations into isomorphic tiling semigroups. Although G_n is evidently an interesting, and well-known, group we have had some difficulty in finding a development of its properties in the literature. Reference [18] describes some of them, as part of a survey concerning connections between different areas of Mathematics under the theme "$2^n\, n!$". In particular, it is outlined the proof that G_n has $2^n\, n!$ members, using an inductive argument, and of a semidirect product decomposition for G_n.

In this appendix, we prove a matrix representation for the symmetries of the n-dimensional hypercube, conclude on the order of this group, and use the matrix representation to give an original proof of the semidirect product decomposition. In addition, we show that, except when $n = 1$, the group G_n is a 2-generator group.

Before proceeding with the matrix representation, we recall some definitions regarding G_n already mentioned in Section 7.1.

As in Section 7.1, we will regard the n-dimensional hypercube to be the set $[-1, 1]^n$, that is

$$H_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon -1 \leqslant x_i \leqslant 1 \text{ for all } i \in [n]\}\ .$$

(Recall that, for each positive integer n, we denote the set $\{1, \ldots, n\}$ by $[n]$.)

The *vertices* of the n-dimensional hypercube are the n-tuples all of whose components are equal either to 1 or -1. Denote by V_n the set of vertices of the hypercube H_n. It is easy to see that V_n is a group under componentwise multiplication, isomorphic to the additive group $\mathbb{Z}_2^n$; from now on, we will regard V_n as such without special mention.

Two vertices $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are said to be *neighbours* if they differ at a single component; that is, if there exists $i \in [n]$ such that $u_i \neq v_i$ and $u_j = v_j$ whenever $j \neq i$. Thus, u and v are neighbours if and only if $u - v$ has exactly one non-zero component, which is either 2 or -2. Of course, each vertex from V_n has n neighbours.

A *symmetry* of the n-dimensional hypercube is a permutation of its vertices that preserves neighbours. The set of symmetries of the n-dimensional hypercube H_n is a group under composition, known as the *full symmetry group of the n-dimensional hypercube* [18] and denoted G_n.

An upper bound for the order of $\mathbf{G}_n$

In order to prove the matrix representation of G_n, and conclude on the order of this group, we begin by establishing an upper bound for the number of its elements. We essentially follow the same strategy as in [18]. The facts used tend to be geometrically convincing, but the details of their proofs are not completely trivial. For that reason, we provide detailed proofs.

The *faces* of the n-dimensional hypercube are the unlabelled, undirected graphs L_i^+ and L_i^-, where $i \in [n]$, with vertex set

$$V(L_i^+) = \{(u_1, \dots, u_n) \in V_n \colon u_i = 1\}$$

and

$$V(L_i^-) = \{(u_1, \dots, u_n) \in V_n \colon u_i = -1\}$$

and set of edges

$$E(L_i^\mu) = \{(u, v) \in V(L_i^\mu) \times V(L_i^\mu) \colon \ u \text{ and } v \text{ are neighbours } \} \ ,$$

for $\mu \in \{-1, 1\}$. There are, of course, $2n$ faces and each face contains 2^{n-1} vertices. Often, we will identify a face with its vertex set. The faces L_i^+ and L_i^- are said to be *opposite faces*, for every $i \in [n]$. Clearly, opposite faces are disjoint sets and, since V_n has 2^n elements, a pair of opposite faces always constitutes a partition of V_n. Each vertex in a face F has $n-1$ neighbours in F and a single neighbour in the opposite face.

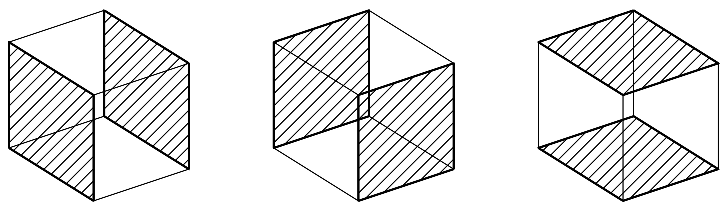


Figure 8.1: The (opposite) faces of H_3

The following fact will be very important. If we identify an hypercube H_n with the graph with vertex set V_n and edges between neighbouring vertices, then, for all $n \geqslant 2$, each face of H_n is isomorphic, as a graph, to H_{n-1}. In fact, consider the map γ from a face $V(L_i^\mu)$ of H_n to V_{n-1} that sends a vertex $(v_1, \dots, v_n)$ to the vertex $(u_1, \dots, u_{n-1})$, where

$$u_j = \begin{cases} v_i, & \text{if } j < i \\ v_{i+1}, & \text{if } j \geqslant i \,. \end{cases}$$

Notice that γ simply forgets the i^{th} component of a vertex from $V(L_i^\mu)$. Thus, it is easy to see that it satisfies the following property: (u, v) is an edge in L_i^μ if and only if $(u\gamma, v\gamma)$ is an edge in H_{n-1}, no matter if $\mu = 1$ or $\mu = -1$, because if w and z are adjacent vertices, then they differ at exactly one component, necessarily other than the i^{th} component, and so also $w\gamma$ and $z\gamma$ differ at exactly one component. The importance of this fact relies on the following consequence: the graph isomorphisms of any face L_i^μ in H_n are in one to one correspondence with the graph isomorphisms of H_{n-1}, that is, the bijections on V_{n-1} that respects neighbouring vertices.

Lemma 1. *Let F be a face of the n-dimensional hypercube H_n and σ a symmetry from G_n. Then $F\sigma$ is a face of the n-dimensional hypercube.*

Proof. For simplicity, we will show that $L_1^+\sigma = L_j^\nu$ for some $j \in [n]$ and $\nu \in \{-1, 1\}$; of course, this conclusion could dually be drawn for L_1^- and analogously for any L_i^+, so that it generalizes to an arbitrary face L_i^μ.

Let $z = (1, \ldots, 1) \in L_1^+$ and $z_1, \ldots, z_n$ be the n neighbours of z, where each z_i has all components equal to 1 except for the i^{th} component, which is -1. Then $z_1 = (-1, 1, \ldots, 1)$ is the only neighbour of z that belongs to $V_n \setminus L_1^+ = L_1^-$ and $z_2, \ldots, z_n$ are the $n-1$ neighbours of z in L_i^μ. Since, by definition of symmetry, $z\sigma$ and $z_1\sigma$ are neighbours, then they must differ at a single component, say j. Taking ν to be the j^{th} component of $z\sigma$, we thus have that the j^{th} component of $z_1\sigma$ is $-\nu$. But then, since $z_2, \ldots, z_n$ are (distinct) neighbours of z but not of z_1, we have that $z_2\sigma, \ldots, z_n\sigma$ are (distinct) neighbours of $z\sigma$ but not of $z_1\sigma$, so that all $n-1$ elements $z_k\sigma$, with $k \in \{2, \ldots, n\}$ must have j^{th} component equal to ν. Thus, $z\sigma, z_2\sigma, \ldots, z_n\sigma \in L_j^\nu$ and $z_1\sigma \in L_j^{-\nu}$.

For $n = 2$, this shows that $L_1^+\sigma \subseteq L_j^\nu$, since L_1^+ has only two elements, namely z and z_2. By duality, also $L_1^-\sigma$ must be contained in some face, and that must be $L_j^{-\nu}$, as $z_1\sigma \in L_j^{-\nu}$. Since σ is a permutation on a finite set, we conclude that $L_1^+\sigma = L_j^\nu$ (and that $L_1^-\sigma = L_j^{-\nu}$).

Now suppose that $n \geqslant 3$; we need to deal with the elements of L_1^+ which are not neighbours of z. We begin by taking an element $y = (y_1, \ldots, y_n) \in L_1^+$ with exactly two components equal to -1 (that is, an element in L_1^+ with two components different from those of z). Then y has two neighbours in common with z in L_1^+. In fact, if y is such that $y_l = y_{l'} = -1$ (where $l, l' \neq 1$) and all other components are equal to 1, then the two neighbours z_l and $z_{l'}$ of z from L_i^μ are also neighbours of y. In order to obtain a contradiction, assume that $y\sigma \in L_j^{-\nu}$. Then $y\sigma$ has $n-1$ neighbours in $L_j^{-\nu}$ and only one neighbour in L_j^ν, a contradiction as all neighbours of z from L_i^+, including u_l and $u_{l'}$, have their images in L_j^ν. Therefore, $y\sigma \in L_j^\nu$. Again, this completes the proof if $n = 3$. Now suppose $n \geqslant 4$ and that $w = (w_1, \ldots, w_n) \in L_1^+$ has three components equal to -1. Then w has three neighbours in L_1^+ (simply replace, one at a time, the -1 components of w by 1). By an argument similar to the one above, we conclude that $w\sigma \in L_j^\nu$. By exhaustion, the same conclusion holds for all elements in L_1^+, no matter how large n is. Also as above, the fact that this shows that the image of L_1^- is contained in some face as well the fact that σ is a permutation on a finite set, allows us to conclude that $L_1^+\sigma = L_j^\nu$. □

Note that the proof above also shows that, if $L_i^\mu\sigma = L_j^\nu$, then $L_i^{-\mu}\sigma = L_j^{-\nu}$. That is, any symmetry σ from G_n maps opposite faces to opposite faces.

Lemma 2. *Any symmetry $\sigma \in G_n$ is determined by its action on a single face.*

Proof. Suppose that $L_1^+\sigma = L_j^\nu$. We claim that the image of any vertex in $V_n \setminus L_1^+$, that is, in L_1^-, is determined by the image of its neighbour in L_1^+. As we just saw, we have $L_1^-\sigma = L_j^{-\nu}$. Let $(1, u_2, \ldots, u_n) \in L_1^+$. Then $(1, u_2, \ldots, u_n)\sigma = (v_1, \ldots v_{j-1}, \nu, v_{j+1}, \ldots, v_n)$, for some $(v_1, \ldots v_{j-1}, \nu, v_{j+1}, \ldots, v_n) \in L_j^\nu$. Since $(1, u_2, \ldots, u_n)$ and $(-1, u_2, \ldots, u_n)$ are neighbours, then $(-1, u_2, \ldots, u_n)\sigma$ is the only neighbour of $(v_1, \ldots v_{j-1}, \nu, v_{j+1}, \ldots, v_n)$ in $L_j^{-\nu}$, namely, $(v_1, \ldots v_{j-1}, -\nu, v_{j+1}, \ldots, v_n)$, as claimed. Similarly, the image of the vertices in any face of H_n is determined by the image of the vertices in the opposite face. □

Proposition 1. *The group G_n has order at most $2^n\, n!$.*

Proof. Since there are $2n$ faces in H_n, there are at most $2n$ possibilities for the image of L_1^+. Assuming that L_j^ν is the image of L_1^+ under a symmetry σ, then $\sigma|_{L_1^+} \colon L_1^+ \to L_j^\nu$ is an isomorphism. Since the graph isomorphisms between faces in H_n are in one to one correspondence with the graph isomorphisms of H_{n-1}, or, equivalently, the symmetries of the $(n-1)$-dimensional hypercube, there are $|G_{n-1}|$ of these. Thus, $|G_n| \leqslant 2n|G_{n-1}|$. Since G_1 has two elements and $2^1\,1! = 2$ and, assuming that $|G_{n-1}| \leqslant 2^{n-1}(n-1)!$, we conclude that

$$|G_n| \leqslant 2n|G_{n-1}| \leqslant 2n\, 2^{n-1}(n-1)! = 2^n\, n!\,,$$

and the result follows by induction. □

Matrix decomposition (and the order of $\mathbf{G}_n$ determined)

Recall from Section 7.1 that a *signed permutation matrix* is an $n \times n$ matrix with exactly one non-zero entry in each column and in each row, which is either 1 or -1. In this section, we prove that G_n, for all non-negative integer n, is isomorphic to the group of $n \times n$ signed permutation matrices (under matrix multiplication).

Let n be a non-negative integer. Denote by $\mathcal{M}_n$ the group of $n \times n$ signed permutation matrices.

Lemma 3. *For each $M \in \mathcal{M}_n$, the mapping $\sigma_M \colon V_n \to V_n$ defined by $u\sigma_M = uM$, for all $u \in V_n$, is a symmetry from G_n.*

Proof. Let $M = (M_{ij}) \in \mathcal{M}_n$.

First, we prove that σ_M sends in fact an element of V_n to an element of V_n. Let $u = (u_1, \ldots, u_n) \in V_n$. By definition of σ_M, we have $u\sigma_M = uM$. Thus,

$$u\sigma_M = \begin{pmatrix} u_1 & \ldots & u_n \end{pmatrix} \begin{pmatrix} M_{11} & \ldots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{n1} & \ldots & M_{nn} \end{pmatrix} = \left(\sum_{i=1}^{n} u_i M_{i1}, \ldots, \sum_{i=1}^{n} u_i M_{in} \right) .$$

Since each column of M contains only one non-zero entry, say $M_{i_j j} \neq 0$ in the j^{th} column, then

$$u\sigma_M = (u_{i_1} M_{i_1 1}, \ldots, u_{i_n} M_{i_n n}) .$$

Moreover, since each u_j is either 1 or -1, as u is a vertex of the hypercube, and $M_{i_j j}$ is either 1 or -1, by definition of signed permutation matrix, we have $u_{i_j} M_{i_j j} = 1$ or $u_{i_j} M_{i_j j} = -1$, for all j. Therefore, $u\sigma_M \in V_n$.

Secondly, we prove that σ_M preserves neighbours; since this implies, in particular, that σ_M is a bijection, we have our conclusion. Let $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ be neighbouring vertices of the n-dimensional hypercube. Then there exists $j \in [n]$ such that $(u-v)_j = \pm 2$ and $(u-v)_k = 0$ for all $k \neq j$. Thus,

$$\begin{aligned} u\sigma_M - v\sigma_M &= uM - vM = (u-v)M \\ &= \left(\sum_{i=1}^{n} (u_i - v_i) M_{i1}, \ldots, \sum_{i=1}^{n} (u_i - v_i) M_{in} \right) \\ &= ((u_j - v_j) M_{j1}, \ldots, (u_j - v_j) M_{jn}) . \end{aligned}$$

But since there is a unique non-zero entry in the j^{th} row of M, say M_{jl}, we get that

$$(u\sigma_M - v\sigma_M)_l = (u_j - v_j) M_{jl} = (\pm 2)\,(\pm 1) = \pm 2$$

and

$$(u\sigma_M - v\sigma_M)_k = (u_j - v_j) M_{jk} = (\pm 2)\, 0 = 0 \,,$$

for all $k \neq l$. Therefore, $u\sigma_M$ and $v\sigma_M$ are neighbours as well. $\square$

Note that different matrices of $\mathcal{M}_n$ define distinct symmetries of G_n. In fact, let M_1 and M_2 be such that $\sigma_{M_1} = \sigma_{M_2}$. We claim that $M_1 = M_2$. Again, consider the vertex $z = (1, \ldots, 1)$ and let $z_1, \ldots, z_n$ be the n neighbours of z, where each z_i has all components equal to 1 except for the i^{th} component, which is -1. Then, each standard basis vector can be written as $e_i = \frac{1}{2}(z - z_i)$, and so

$$e_i(M_1 - M_2) = e_i M_1 - e_i M_2 = e_i \sigma_{M_1} - e_i \sigma_{M_2} = \frac{1}{2}(z - z_i)\sigma_{M_1} - \frac{1}{2}(z - z_i)\sigma_{M_2} = 0 \,,$$

since $\sigma_{M_1} = \sigma_{M_2}$. But then $M_1 = M_2$, as claimed.

Consequently, $|\mathcal{M}_n| \leqslant |G_n|$, as the map $\Phi \colon \mathcal{M}_n \to G_n$ that sends each $n \times n$ signed permutation matrix M to the symmetry σ_M is injective. Since there exist $2^n\, n!$ signed permutation matrices, we conclude from Proposition 1 that

Proposition 2. *The full symmetry group G_n of symmetries of the n-dimensional hypercube has order $2^n\, n!$.*

In particular, Φ is a bijection from the group $\mathcal{M}_n$ to the full symmetry group G_n. The next result shows that Φ is a homomorphism as well:

Lemma 4. *Let $M_1, M_2 \in \mathcal{M}_n$. Then $\sigma_{M_1M_2} = \sigma_{M_1}\sigma_{M_2}$.*

Proof. Let $M_1, M_2 \in \mathcal{M}_n$ and let $u \in V_n$. We claim that $u\sigma_{M_1M_2} = u(\sigma_{M_1}\sigma_{M_2})$. In fact,

$$u\sigma_{M_1M_2} = u(M_1M_2) = (uM_1)M_2 = (uM_1)\sigma_{M_2} = (u\sigma_{M_1})\sigma_{M_2} = u(\sigma_{M_1}\sigma_{M_2})\,,$$

as claimed. $\square$

In conclusion,

Theorem 1. *For each non-negative integer n, the full symmetry group G_n is isomorphic to the group $\mathcal{M}_n$ of $n \times n$ signed permutation matrices.*

This result can also be used to show a fact that was very important in Chapter 7.

Proposition 3. *Each symmetry of G_n extends to an automorphism of $\mathbb{Z}^n$ which is represented by a signed permutation matrix. Conversely, each signed permutation matrix is the matrix of some automorphism of $\mathbb{Z}^n$ which restricts to a symmetry from G_n.*

Proof. As we have seen, the map $\Phi\colon \mathcal{M}_n \to G_n$ defined by $M\Phi = \sigma_M$, where $u\sigma_M = uM$ for all $u \in V_n$, is an isomorphism. Then Φ is invertible, and, for each $\sigma \in G_n$, we have that $\sigma\Phi^{-1} = M_\sigma$ is the matrix such that $M_\sigma\Phi = \sigma$, that is, $u\sigma = uM_\sigma$ for all $u \in V_n$.

Let $\sigma \in G_n$. Consider the matrix $M_\sigma = \sigma\Phi^{-1}$. Since M_σ is a signed permutation matrix, we have that $\det M_\sigma$ is either 1 or -1, and so the mapping φ defined by $x\varphi = xM_\sigma$, with $x \in \mathbb{Z}^n$, is an automorphism of $\mathbb{Z}^n$. But then, for all $u \in V_n$, we have $u\sigma = uM_\sigma = u\varphi$, that is, $\varphi|_{V_n} = \sigma$. Therefore, σ extends to an automorphism of $\mathbb{Z}^n$. Notice that, for all $i \in [n]$,

$$e_i\varphi = e_iM_\sigma = \frac{1}{2}(z - z_i)M_\sigma = \frac{1}{2}(zM_\sigma - z_iM_\sigma) = \frac{1}{2}(z\sigma - z_i\sigma)$$

(where $z = (1, \ldots, 1)$ and $z_1, \ldots, z_n$ be the n neighbours of z).

For the converse, let $M \in \mathcal{M}_n$. We know that the mapping φ defined by $x\varphi = xM$, with $x \in \mathbb{Z}^n$, is an automorphism of $\mathbb{Z}^n$. Then, by Lemma 3, $\varphi|_{V_n} = \sigma_M$ is a symmetry of the n-dimensional hypercube. $\square$

A semidirect product decomposition

In this section, we give a proof of a semidirect product decomposition of G_n. In view of the previous theorem, it suffices to prove a semidirect product decomposition of $\mathcal{M}_n$.

We begin by recalling the notion of semidirect product of groups. Let H and A be groups and $\alpha\colon H \to \mathrm{Aut}A$ a group homomorphism from H into the group of automorphisms of A. For $g \in H$ and $a \in A$, denote $a(g\alpha)$, the image of a under $g\alpha$, by a^g. The fact that α is a group homomorphism implies that $a^{gh} = (a^g)^h$ and $a^{1_H} = a$, and also that $(ab)^g = a^g b^g$, as each $g\alpha$ is an automorphism of A, for all $a, b \in A$ and $g, h \in H$. Actually, these conditions are equivalent to α being a group homomorphism of H into $\mathrm{Aut}A$ (we say that H *acts (on the right) on* A). This allows to define the *semidirect product* $H \rtimes_\alpha A$ *of* H *and* A *with respect to the group homomorphism* α as the group with element set $H \times A$ and operation defined by

$$(g, a)(h, b) = (gh, a^h b)\,,$$

for all $(g, a), (h, b) \in H \times A$. Often, $H \rtimes_\alpha A$ is denoted simply by $H \rtimes A$. The identity of $H \rtimes A$ is $(1_H, 1_A)$ and the inverse of an element (g, a) is $\left(g^{-1}, (a^{-1})^{g^{-1}}\right)$. It is easy to see where does the term "semidirect product" comes from: when $a^g = a$, for all $a \in A$ and $g \in H$, then $H \rtimes A$ reduces to the direct product $H \times A$.

Because the maps $a \mapsto (1_H, a)$ from A to $H \rtimes A$ and $g \mapsto (g, 1_A)$ from H to $H \rtimes A$ are embeddings, the groups A and H can be identified with their respective images in $H \rtimes A$. With this identification, A is a normal subgroup of $H \rtimes A$ and H is a subgroup of $H \rtimes A$ with $A \cap H$ the trivial group and $HA = H \rtimes A$. In fact, if G is a group and H and A are subgroups of G such that A is a normal subgroup of G with $A \cap H = \{1_G\}$ and $G = HA$, then G is isomorphic to a semidirect product of H and A, namely the one associated with the group homomorphism $\alpha\colon H \to \mathrm{Aut}A$ defined by $a^g = g^{-1}ag$, for all $a \in A$ and $g \in H$ (see, for example, [22]).

We now prove that, for all non-negative integer n, the group $\mathcal{M}_n$ is isomorphic to a semidirect product of S_n and $\mathbb{Z}_2^n$.

Let n be a non-negative integer. Consider the following subgroups of $\mathcal{M}_n$: the group H of $n \times n$ *permutation matrices*, that is, the $n \times n$ matrices with exactly one non-zero entry in each column and in each row, which is equal to 1, and the group A of *signed diagonal matrices*, that is, the diagonal matrices with all non-zero entries equal to 1 or -1.

It is easy to see that the group H is isomorphic to the symmetric group S_n, the group A is isomorphic to $\mathbb{Z}_2^n$, and that $A \cap H = \{I_n\}$.

Lemma 5. *The map* $\theta\colon \mathcal{M}_n \to \mathcal{M}_n$ *that sends each signed permutation matrix* $M = (M_{ij})$ *to the corresponding (unsigned) permutation matrix, that is, to the matrix with* $(M\theta)_{ij} = |M_{ij}|$, *for all* $i, j \in [n]$, *is an endomorphism of* $\mathcal{M}_n$ *with kernel* A.

Proof. Clearly, the signed permutation matrices which are sent under θ to the identity matrix I_n are the signed diagonal matrices, so that $\ker\theta = A$.

Let $M, N \in \mathcal{M}_n$. Since M and N are signed permutation matrices, we have

$$\begin{aligned}
(M\theta\, N\theta)_{ij} &= \sum_{k=1}^{n} (M\theta)_{ik}\, (N\theta)_{kj} \\
&= \sum_{k=1}^{n} |M_{ik}|\, |N_{kj}| \\
&= \begin{cases} |M_{ik_{ij}}|\, |N_{k_{ij}j}| & \text{if there exists } k_{ij} \text{ such that } M_{ik_{ij}} \neq 0 \text{ and } N_{k_{ij}j} \neq 0 \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

for all $i, j \in [n]$. On the other hand,

$$\begin{aligned}
((M\, N)\theta)_{ij} &= \left(\left(\sum_{l=1}^{n} M_{il}\, N_{lj} \right) \theta \right)_{ij} \\
&= \left| \sum_{l=1}^{n} M_{il}\, N_{lj} \right| \\
&= \begin{cases} |M_{il_{ij}}\, N_{l_{ij}j}| & \text{if there exists } l_{ij} \text{ such that } M_{il_{ij}} \neq 0 \text{ and } N_{l_{ij}j} \neq 0 \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

again because M and N are signed permutation matrices, for all $i, j \in [n]$. Since, for each $i, j \in [n]$, we must have $l_{ij} = k_{ij}$ and $|M_{ik_{ij}}|\,|N_{k_{ij}j}| = |M_{ik_{ij}}\, N_{k_{ij}j}|$, we conclude that, for all $i, j \in [n]$, we have $(M\theta\, N\theta)_{ij} = ((M\, N)\theta)_{ij}$. Therefore, $(M\, N)\theta = M\theta\, N\theta$. □

Being the kernel of an endomorphism, A is a normal subgroup of $\mathcal{M}_n$. Also,

Lemma 6. *$\mathcal{M}_n = HA$.*

Proof. Clearly, $HA \subseteq \mathcal{M}_n$. Conversely, let $M = (M_{ij}) \in \mathcal{M}_n$. Consider the matrices $P = M\theta$, with θ as in the previous lemma, and $D = (D_{ij})$ with

$$D_{jj} = \begin{cases} 1 & \text{if there exists } i \text{ such that } M_{ij} = 1 \\ -1 & \text{if there exists } i \text{ such that } M_{ij} = -1 \end{cases}$$

and $D_{ij} = 0$ for all $j \neq i$. Clearly, $P \in H$ and $D \in A$. Since, for all $i, j \in [n]$, we have

$$\begin{aligned}
(PD)_{ij} &= \sum_{k=1}^{n} P_{ik}\, D_{kj} \\
&= P_{ij}\, D_{jj} && \text{(as } D_{kj} = 0 \text{ for all } k \neq j\text{)} \\
&= |M_{ij}|\, D_{jj} \\
&= M_{ij}
\end{aligned}$$

as $|M_{ij}| \neq 0$ if and only if $M_{ij} \neq 0$, in which case $|M_{ij}|\, D_{jj} = 1\, D_{jj} = M_{ij}$, by definition of D. We conclude that $M = PD \in HA$. □

Putting together these last results, we have

Theorem 2. *For each non-negative integer n, the group $\mathcal{M}_n$ of signed permutation matrices is isomorphic to a semidirect product of S_n and $\mathbb{Z}_2^n$, where S_n acts on $\mathbb{Z}_2^n$.*

In view of Theorems 1 and 2, we obtain,

Corollary 1. *For each non-negative integer n, the full symmetry group G_n of the n-dimensional hypercube is isomorphic to a semidirect product of S_n and $\mathbb{Z}_2^n$.*

Next, we aim at explicitly describing the isomorphism from G_n to $S_n \rtimes \mathbb{Z}_2^n$. As mention before, a symmetry σ from G_n can be identified with the matrix M associated, with respect to the standard basis, with the automorphism of $\mathbb{Z}^n$ that extends σ. Also, considering the vector $z = (1, \ldots, 1)$ and its n neighbours $z_1, \ldots, z_n$, where each z_i has all components equal to 1 except for the i^{th} component, which is -1, we have $e_i = \frac{1}{2}(z - z_i)$, for all $i \in [n]$, and so

$$M = \begin{pmatrix} e_1\sigma \\ \vdots \\ e_n\sigma \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(z\sigma - z_1\sigma) \\ \vdots \\ \frac{1}{2}(z\sigma - z_1\sigma) \end{pmatrix}.$$

Thus, in order to fully describe the isomorphism between G_n and $S_n \rtimes \mathbb{Z}_2^n$, we now investigate the correspondence between $\mathcal{M}_n$ and $S_n \rtimes \mathbb{Z}_2^n$.

Let $M \in \mathcal{M}_n$. Then $M = PD$ with P a permutation matrix and D a signed diagonal matrix as in Lemma 6. Notice that P and D are completely determined by M, and, thus, (P, D) is the unique pair in $H \times A$ that corresponds to M in this way. On the other hand, such a pair $(P, D) \in H \times A$ corresponds to a (unique) pair $(\tau, u) \in S_n \times \mathbb{Z}_2^n$, since $H \simeq S_n$ and $A \simeq \mathbb{Z}_2^n$. More precisely, we have that τ is the permutation on $[n]$ defined by $(i)\tau = j$ if and only if $(P_{i1}, \ldots, P_{in}) = e_j$ and u is the vector with i^{th} component equal to D_{ii}.

Example 1. In $\mathbb{R}^3$, fix the following enumeration of the vertices of the hypercube:

$$\begin{array}{llll} \mathbf{1} = (1,1,1) & \mathbf{2} = (1,-1,1) & \mathbf{3} = (-1,-1,1) & \mathbf{4} = (-1,1,1) \\ \mathbf{5} = (1,1,-1) & \mathbf{6} = (1,-1,-1) & \mathbf{7} = (-1,-1,-1) & \mathbf{8} = (-1,1,-1) \end{array}$$

and assume that $\sigma \in G_n$ is given by

$$\sigma = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} \\ \mathbf{3} & \mathbf{2} & \mathbf{6} & \mathbf{7} & \mathbf{4} & \mathbf{1} & \mathbf{5} & \mathbf{8} \end{pmatrix}.$$

Since

$$e_1 = (1,0,0) = \frac{1}{2}(\mathbf{1} - \mathbf{4}), \quad e_2 = (0,1,0) = \frac{1}{2}(\mathbf{1} - \mathbf{2}), \text{ and } e_3 = (0,0,1) = \frac{1}{2}(\mathbf{1} - \mathbf{5}),$$

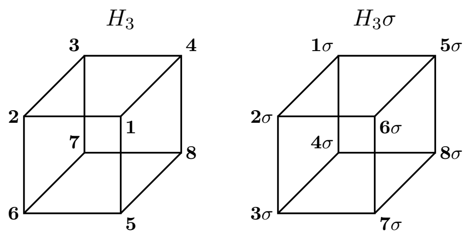


Figure 8.2: H_3 under σ

we have

$$\begin{aligned}
e_1\sigma &= \frac{1}{2}(\mathbf{1}\sigma - \mathbf{4}\sigma) = \frac{1}{2}(\mathbf{3} - \mathbf{7}) = \frac{1}{2}(0,0,2) = (0,0,1)\\
e_2\sigma &= \frac{1}{2}(\mathbf{1}\sigma - \mathbf{2}\sigma) = \frac{1}{2}(\mathbf{3} - \mathbf{2}) = \frac{1}{2}(-2,0,0) = (-1,0,0)\\
e_3\sigma &= \frac{1}{2}(\mathbf{1}\sigma - \mathbf{5}\sigma) = \frac{1}{2}(\mathbf{3} - \mathbf{1}) = \frac{1}{2}(0,-2,0) = (0,-1,0)\,.
\end{aligned}$$

Thus, the matrix associated with σ is

$$M = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

Since $M = PD$ with

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we conclude that M, and therefore σ, corresponds to the pair (τ, u) given by

$$\tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \text{and} \quad u = (-1,-1,1)\,.$$

We end this section with the description of the action of S_n on $\mathbb{Z}_2^n$. Of course, if $(P, D) \in H \times A$ corresponds to $(\sigma, u) \in S_n \times \mathbb{Z}_2^n$ and $(Q, C) \in H \times A$ corresponds to $(\rho, v) \in S_n \times \mathbb{Z}_2^n$, then $(P, D)(Q, C)$ must correspond to $(\sigma, u)(\rho, v)$, that is, $(PQ, D^Q C)$ must correspond to $(\sigma\rho, u^\rho v)$. Since $D^Q = Q^{-1}DQ$, then, if $E \in A$ is the signed diagonal matrix associated with u^ρ, we have $E = Q^{-1}DQ$, or, equivalently, $QE = DQ$. Thus, $(QE)_{ij} = (DQ)_{ij}$, for all $i, j \in [n]$. Since D and E are diagonal matrices, we have

$$(QE)_{ij} = \sum_{k=1}^{n} Q_{ik}\, E_{kj} = Q_{ij}\, E_{jj}$$

and

$$(DQ)_{ij} = \sum_{k=1}^{n} D_{ik}\, Q_{kj} = D_{ii}\, Q_{ij}\,.$$

Notice that $(QE)_{ij}$ and $(DQ)_{ij}$ are non-zero if and only if $Q_{ij} \neq 0$, in which case $Q_{ij} = 1$, as Q is a permutation matrix. Then $Q_{ij} \neq 0$ implies that

$$E_{jj} = Q_{ij}\, E_{jj} = (QE)_{ij} = (DQ)_{ij} = D_{ii}\, Q_{ij} = D_{ii}\,.$$

But since $Q_{ij} \neq 0$ if and only if $i\sigma = j$, we conclude that

$$E_{jj} = D_{ii} = u_i = u_{j\sigma^{-1}}\,.$$

Therefore, $u^\sigma = (u_{1\sigma^{-1}}, \ldots, u_{n\sigma^{-1}})$. Hence,

$$(\sigma, u)(\rho, v) = (\sigma\rho, u^\rho v) = (\sigma\rho, (u_{1\sigma^{-1}}, \ldots, u_{n\sigma^{-1}})(v_1, \ldots, v_n))\,,$$

is the operation defined on $S_n \rtimes \mathbb{Z}_2^n$, for all $(\sigma, u), (\rho, v) \in S_n \times \mathbb{Z}_2^n$.

A minimal generating set

As noted in Chapter 7, Section 7.3, the group G_1 has only two elements, the identity element and the reflection about the origin, and is, thus, generated as a group by a single element. It is not hard to see that G_2 is generated, as a group, by any reflection and any rotation. For example, taking $\rho = \sigma_x$ and $\omega = \sigma_{\frac{\pi}{2}}$ (in the notation of Section 7.1), we have:

$$id = \rho^2,\ \sigma_y = \rho\omega^2,\ \sigma_{y=x} = \rho\omega,\ \sigma_{y=-x} = \rho\omega^3,\ \sigma_\pi = \omega^2,\ \text{and } \sigma_{\frac{3\pi}{2}} = \omega^3\,.$$

Less obvious is that each G_n, with $n \geqslant 2$, is generated by two elements as well. To prove this, we will use the decomposition of G_n obtained above. Recall that, identifying S_n and $\mathbb{Z}_2^n$ with the subgroups $S_n \times \{1\}$ and $\{\iota\} \times \mathbb{Z}_2^n$ of $S_n \rtimes \mathbb{Z}_2^n$, respectively, we have that $S_n \rtimes \mathbb{Z}_2^n = S_n\, \mathbb{Z}_2^n$.

Let $\tau = (1, 2) \in S_n$ and $u = (-1, 1, \ldots, 1) \in \mathbb{Z}_2^n$ and consider the elements $g_1 = (\tau, u)$ and $g_2 = (\phi, u)$ of $S_n \rtimes \mathbb{Z}_2^n$, where ϕ will depend on n being even or odd. We claim that, in either case, $S_n \rtimes \mathbb{Z}_2^n = \langle g_1, g_2 \rangle$, where, of course, $\langle g_1, g_2 \rangle$ denotes the subgroup of $S_n \rtimes \mathbb{Z}_2^n$ generated by g_1 and g_2. Clearly, $\langle g_1, g_2 \rangle \subseteq S_n \rtimes \mathbb{Z}_2^n$; it remains to show the converse inclusion.

Let

$$S = \{\rho \in S_n \colon (\rho, (1, \ldots, 1)) \in \langle g_1, g_2 \rangle\}$$

and

$$Z = \{v \in \mathbb{Z}_2^n \colon (\iota, v) \in \langle g_1, g_2 \rangle\}\ .$$

It is straightforward to check that S is a subgroup of S_n and that Z is a subgroup of $\mathbb{Z}_2^n$. Moreover, by definition, $SZ \subseteq \langle g_1, g_2 \rangle$. We claim that $S_n \rtimes \mathbb{Z}_2^n = S_n\, \mathbb{Z}_2^n = SZ$; this will prove that $S_n \rtimes \mathbb{Z}_2^n \subseteq \langle g_1, g_2 \rangle$. Both in the even and odd cases, our strategy is the following: on the one hand, we prove that S_n is generated by τ and ϕ and that these permutations belong to S, so that $S = S_n$; on the other hand, we prove that each "standard" basis vector of $\mathbb{Z}_2^n$, namely the n-tuples with a single entry equal to -1, belongs to Z, so that $Z = \mathbb{Z}_2^n$.

Even case. Let $\phi = (2, 3, \ldots, n)$. Since $(2, 3, \ldots, n)^{-1} = (2, n, n-1, \ldots, 3)$, we have

$$
\begin{aligned}
(2, 3, \ldots, n)^{-1}(1, 2)(2, 3, \ldots, n) &= (1, 3)\\
(2, 3, \ldots, n)^{-1}(1, 3)(2, 3, \ldots, n) &= (1, 4)\\
&\vdots\\
(2, 3, \ldots, n)^{-1}(1, n-1)(2, 3, \ldots, n) &= (1, n)\,,
\end{aligned}
$$

and so $(1, i) \in \langle \tau, \phi \rangle$, for all $i \in [n]$. But then $(1, 2, \ldots, n) = (1, 2)(1, 3) \ldots (1, n) \in \langle \tau, \phi \rangle$ and, thus, $S_n = \langle \tau, (1, 2, \ldots, n) \rangle \subseteq \langle \tau, \phi \rangle$. Therefore, $S_n = \langle \tau, \phi \rangle$, that is, the permutations τ and ϕ generate S_n.

Since $1\phi = 1$, we have $u^\phi = (-1, 1, \ldots, 1) = u$. Thus,

$$
\begin{aligned}
g_2^2 &= (\phi^2, u^\phi u) = (\phi^2, u^2) = (\phi^2, (1, \ldots, 1))\,,\\
g_2^3 &= (\phi^3, u^3) = (\phi^3, u)\,,\\
g_2^4 &= (\phi^4, u^4) = (\phi^4, (1, \ldots, 1))\,,\\
g_2^5 &= (\phi^5, u^5) = (\phi^5, u)\,,
\end{aligned}
$$

and, in general,

$$
{g_2}^r = \begin{cases} (\phi^r, (1, \ldots, 1)), & \text{if } r \text{ is even}\\ (\phi^r, u), & \text{if } r \text{ is odd.} \end{cases}
$$

In particular, since n is even, ${g_2}^n = (\phi^n, (1, \ldots, 1)) = (\phi, (1, \ldots, 1))$ implies that $\phi \in S$, and ${g_2}^{n-1} = (\phi^{n-1}, u) = (\iota, u)$, as in this case $n - 1$ is odd, implies that $u = (-1, 1, \ldots, 1) \in Z$. Consequently, $(\tau, u)(\iota, u) = (\tau, u^2) = (\tau, (1, \ldots, 1))$ implies that $\tau \in S$. Therefore, $S_n = \langle \tau, \phi \rangle \subseteq S$, as τ and ϕ generate S_n.

But then, we have that, in particular, the n-cycle $c = (1, 2, \ldots, n)$ belongs to S, so that

$$
(\iota, (1, -1, 1, \ldots, 1)) = (c, (1, \ldots, 1))^{-1}(\iota, u)(c, (1, \ldots, 1)) \in \langle g_1, g_2 \rangle\ ,
$$

as

$$
\begin{aligned}
(c, (1, \ldots, 1))^{-1} &= \left(c^{-1}, ((1, \ldots, 1)^{-1})^{c^{-1}}\right)\\
&= \left((1, n, n-1, \ldots, 3, 2), (1, \ldots, 1)^{(1,n,n-1,\ldots,3,2)}\right)\\
&= ((1, n, n-1, \ldots, 3, 2), (1, \ldots, 1))\,.
\end{aligned}
$$

Thus, $(1, -1, 1, \ldots, 1) \in Z$. It follows that

$$
(\iota, (1, 1, -1, 1, \ldots, 1)) = (c, (1, \ldots, 1))^{-1}(\iota, (1, -1, 1, \ldots, 1))(c, (1, \ldots, 1)) \in \langle g_1, g_2 \rangle\ ,
$$

and so $(1, 1, -1, 1, \ldots, 1) \in Z$ as well. Iterating, we conclude that all n-tuples of $\mathbb{Z}_2^n$ with a single entry equal to -1 belong to Z. Therefore, $\langle Z \rangle = \mathbb{Z}_2^n$ and, hence, $\langle g_1, g_2 \rangle = SZ = S_n\mathbb{Z}_2^n$.

Odd case. Now let $\phi = (1, 2, \ldots, n)$; it is well-known that τ and ϕ generate S_n. Note that $\phi^{-1} = (1, n, n-1, \ldots, 3, 2)$, so that

$$\begin{aligned}(v_1, v_2, \ldots, v_n)^{\phi} &= (v_{1\phi^{-1}}, v_{2\phi^{-1}}, \ldots, v_{n\phi^{-1}}) = (v_n, v_1, v_2, \ldots, v_{n-1}) \quad \text{and} \\ (v_1, v_2, \ldots, v_n)^{\phi^{-1}} &= (v_{1\phi}, v_{2\phi}, \ldots, v_{n\phi}) = (v_2, v_3, \ldots, v_n, v_1)\,,\end{aligned}$$

for all $(v_1, v_2, \ldots, v_n) \in \mathbb{Z}_2^n$.

First, we have that, as in the even case, $\tau \in S$ as $(\tau, (1, \ldots, 1)) = (\tau, u)(\iota, u) \in \langle g_1, g_2\rangle$.

Next, since

$${g_1}^2 = (\tau, u)(\tau, u) = (\tau^2, u^\tau u) = (\iota, (1, -1, 1, \ldots, 1)(-1, 1, \ldots, 1)) = (\iota, (-1, -1, 1, \ldots, 1))\,,$$

then $(-1, -1, 1, \ldots, 1) \in Z$; as

$$\begin{aligned}(\phi, u)^{-1} &= \big(\phi^{-1}, ((-1, 1, \ldots, 1)^{-1})^{\phi^{-1}}\big) \\ &= \big(\phi^{-1}, (-1, 1, \ldots, 1)^{\phi^{-1}}\big) \\ &= (\phi^{-1}, (1, \ldots, 1, -1))\,,\end{aligned}$$

we have, further, that

$$\begin{aligned}{g_2}^{-1}(\iota, (-1, -1, 1, \ldots, 1))g_2 &= (\phi, u)^{-1}(\iota, (-1, -1, 1, \ldots, 1))(\phi, u) \\ &= (\phi^{-1}, (1, \ldots, 1, -1))(\iota, (-1, -1, 1, \ldots, 1))(\phi, u) \\ &= (\phi^{-1}, (1, \ldots, 1, -1)(-1, -1, 1, \ldots, 1))(\phi, u) \\ &= (\phi^{-1}, (-1, -1, 1, \ldots, 1, -1))(\phi, u) \\ &= (\iota, (-1, -1, 1, \ldots, 1, -1)^{\phi}(-1, 1, \ldots, 1)) \\ &= (\iota, (-1, -1, -1, 1, \ldots, 1)(-1, 1, \ldots, 1)) \\ &= (\iota, (1, -1, -1, 1, \ldots, 1))\end{aligned}$$

belongs to $\langle g_1, g_2\rangle$, which implies that $(1, -1, -1, 1, \ldots, 1) \in Z$. By repeatedly conjugating as above, we conclude that each $\varepsilon_i \in \mathbb{Z}_2^n$ with all entries except the i^{th} and the $(i-1)^{\text{th}}$ equal to 1 belongs to Z, for all $i \in \{1, \ldots, n-1\}$.

On the other hand,

$$\begin{aligned}{g_2}^2 &= (\phi^2, u^\phi u) = (\phi^2, (1, -1, 1, \ldots, 1)(-1, 1, \ldots, 1)) = (\phi^2, (-1, -1, 1, \ldots, 1)) \\ {g_2}^3 &= (\phi^3, (-1, -1, 1, \ldots, 1)^\phi u) = (\phi^3, (1, -1, -1, 1, \ldots, 1)u) = (\phi^3, (-1, -1, -1, 1, \ldots, 1)) \\ {g_2}^4 &= (\phi^4, (-1, -1, -1, -1, 1, \ldots, 1)) \\ &\ \vdots \\ {g_2}^n &= (\phi^n, (-1, -1, \ldots, -1)) = (\iota, (-1, -1, \ldots, -1))\,.\end{aligned}$$

Therefore, $(-1,-1,\dots,-1) \in Z$. Since n odd implies that $n-1$ is even, we have

$$\begin{aligned}
&(-1,-1,\dots,-1)\varepsilon_2\varepsilon_4\dots\varepsilon_{n-1} = \\
&\quad = (-1,-1,\dots,-1)(1,-1,-1,1,\dots,1)(1,1,1,-1,-1,1,\dots,1)\dots(1,\dots,1,-1,-1) \\
&\quad = (-1,1,1,-1,\dots,-1)(1,1,1,-1,-1,1,\dots,1)\dots(1,\dots,1,-1,-1) \\
&\quad = (-1,1,\dots,1)\,,
\end{aligned}$$

and so $u = (-1,1,\dots,1) \in Z$. But then $(\phi,(1,\dots,1)) = (\phi,u)(\iota,u) \in \langle g_1,g_2\rangle$, so that $\phi \in S$. Thus, $S_n = \langle \tau,\phi\rangle = S$.

Finally,

$$\begin{aligned}
(\phi,u)^{-1}(\iota,(-1,1,\dots,1))(\phi,u) &= (\phi^{-1},(1,\dots,1,-1))(\iota,(-1,1,\dots,1))(\phi,u) \\
&= (\phi^{-1},(1,\dots,1,-1)(-1,1,\dots,1))(\phi,u) \\
&= (\phi^{-1},(-1,1,\dots,1,-1))(\phi,u) \\
&= (\iota,(-1,-1,1,\dots,1)(-1,1,\dots,1)) \\
&= (\iota,(1,-1,1,\dots,1))
\end{aligned}$$

implies that $(1,-1,1,\dots,1) \in Z$,

$$\begin{aligned}
(\phi,u)^{-1}(\iota,(1,-1,1,\dots,1))(\phi,u) &= (\phi^{-1},(1,\dots,1,-1))(\iota,(1,-1,1,\dots,1))(\phi,u) \\
&= (\phi^{-1},(1,\dots,1,-1)(1,-1,1,\dots,1))(\phi,u) \\
&= (\phi^{-1},(1,-1,1,\dots,1,-1))(\phi,u) \\
&= (\iota,(-1,1,-1,1,\dots,1)(-1,1,\dots,1)) \\
&= (\iota,(1,1,-1,1,\dots,1))
\end{aligned}$$

implies that $(1,1,-1,1,\dots,1) \in Z$, and, analogously, again every n-tuple of $\mathbb{Z}_2^n$ with a single entry equal to -1 belongs to Z. Therefore, $\langle Z\rangle = \mathbb{Z}_2^n$. Hence, $\langle g_1,g_2\rangle = SZ = S_n\mathbb{Z}_2^n$.

In conclusion,

Theorem 3. *Let $n \geqslant 2$. Consider the elements $u \in \mathbb{Z}_2^n$ and $\tau,\phi \in S_n$:*

$$u = (-1,1,\dots,1), \quad \tau = (1,2), \quad \phi = \begin{cases} (2,3,\dots,n), & \text{if } n \text{ is even} \\ (1,2,\dots,n), & \text{if } n \text{ is odd.} \end{cases}$$

Let $g_1 = (\tau,u)$ and $g_2 = (\phi,u)$. Then $S_n \rtimes \mathbb{Z}_2^n = \langle g_1,g_2\rangle$.

In view of Corollary 1, we conclude the following

Corollary 2. *For all $n \geqslant 2$, the group G_n of symmetries of the n-dimensional hypercube is generated, as a group, by two elements.*

Some open questions

We close this work with a record of some open questions, that come either from uncompleted investigations or have been motivated by the investigation, but by their nature do not fit in the scope of the present text, and that constitute possible future work.

Question 1. In Example 2.3.4, we saw how naturally a Bruck-Reilly extension of a monoid can be obtained as a generalized Bruck-Reilly extension of a semigroup by an anti-homomorphism. Using their respective definitions, Dombi and Gilbert in [13] as well as Yamamura in [60] showed, independently, that Bruck-Reilly extensions of monoids can be obtained as HNN extensions of (distinct) inverse monoids. What is the relation between HNN extensions and generalized Bruck-Reilly extensions? Is it possible to retrieve Dombi and Gilbert's and Yamamura's results from this connection? More generally, are there other semigroup constructions which can be represented using generalized Bruck-Reilly extensions?

Question 2. The notion of language of an n-dimensional hypercubic tiling introduced in Section 4.1 (cf. Definition 4.1.3) allowed for a generalization of the language representation of a one-dimensional tiling semigroup. Given that this generalization basically consists in an operation defined on some particular subgraphs of a "coloured" Cayley graph of $\mathbb{Z}^n$, is it possible to further generalize it to arbitrary n-dimensional tilings, possibly making use of a different, more suitable group?

Question 3. In Section 4.1, we defined the notion of n-dimensional factorial language (cf. Definition 4.1.5). What are the language-theoretic properties of this class of languages?

Question 4. In Section 4.3, we saw how every n-dimensional hypercubic tiling semigroup can be constructed as a Rees factor semigroup of an inverse subsemigroup of generalized Bruck-Reilly extension of a semigroup by an anti-homomorphism (Theorem 4.3.10). Moreover, the group G, the semigroup T, and the anti-homomorphism θ are fairly easy to guess from the tiling. The freedom in the choice of these ingredients and how general they can be suggests that this may be a good approach to tiling semigroups of arbitrary tilings, where the wide range of possibilities has made further investigations almost impossible to carry on. It is therefore natural to ask if it is possible to generalize Theorem 4.3.10 to arbitrary tilings.

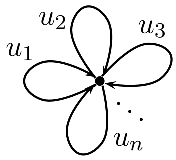
Question 5. In Chapter 6, Section 6.1 we were able to prove that all periodic languages have finitely presented inverse semigroup and we also characterized the ultimately periodic languages satisfying that property. It is possible to complete this characterization to include all languages of factors of bi-infinite words? In particular, and according to [3], a language L over an alphabet Σ is the language of factors of a bi-infinite word over Σ if and only if

1) L is factorial;

2) L is extensible;

3) for all $u, v \in L$ there exists $w \in L$ such that $u, v \in F(w)$.

Thus, for example, if $u_1, u_2, \ldots, u_n \in \Sigma^+$, then the language recognized by the flower automaton



in which all states are both initial and final (and where, for simplicity, we have omitted all states joining consecutive letters of the words $u_1, u_2, \ldots, u_n$), belongs to the class of languages of factors of bi-infinite words. Examples 6.1.2 and 6.1.11 show that the inverse semigroup associated with such a language can be finite or infinitely presented. Is there a general characterization for this case?

Question 6. In Section 6.2, it is shown that n-dimensional hypercubic tiling semigroups are infinitely presented even as strongly E^*-unitary inverse semigroups. Can a (non-trivial) presentation be found for these semigroups?

Question 7. As part of the program proposed by Belissard, as mentioned in the Introduction, Kellendonk aims at studying a C^*-algebra associated with the tiling that models the solid, for which he uses a groupoid constructed from the tiling semigroup in a standard way: it consists of the minimal elements among the equivalence classes of all decreasing sequences of elements from the semigroup, for a certain equivalence relation. Equipped with a suitable topology, the groupoid turns into a topological groupoid. As also mentioned in the Introduction, in [34] Lenz gives an alternative construction. He considers the set of equivalence classes of a certain equivalence relation defined on the set of down directed subsets of the semigroup (isomorphic to Paterson's universal groupoid) and then its subgroupoid consisting of its minimal elements, equipped with the induced topology.

It is possible to give yet another description of Kellendonk's topological groupoid of a tiling, which resembles Lenz's version in that it uses down directed sets as building blocks, but differs in avoiding the need to consider an equivalence relation. In fact, it can be proved that Kellendonk's groupoid is isomorphic to the set consisting of all proper ultrafilters on the tiling semigroup equipped with the operation defined by

$$U * V = \{x \in S \colon x \geqslant uv \text{ for some } u \in U \text{ and } v \in V\}\ .$$

Considering on this set the appropriate topology, we have that this topological groupoid is isomorphic to Kellendonk's topological groupoid. (Cf. Figure 9.1.)

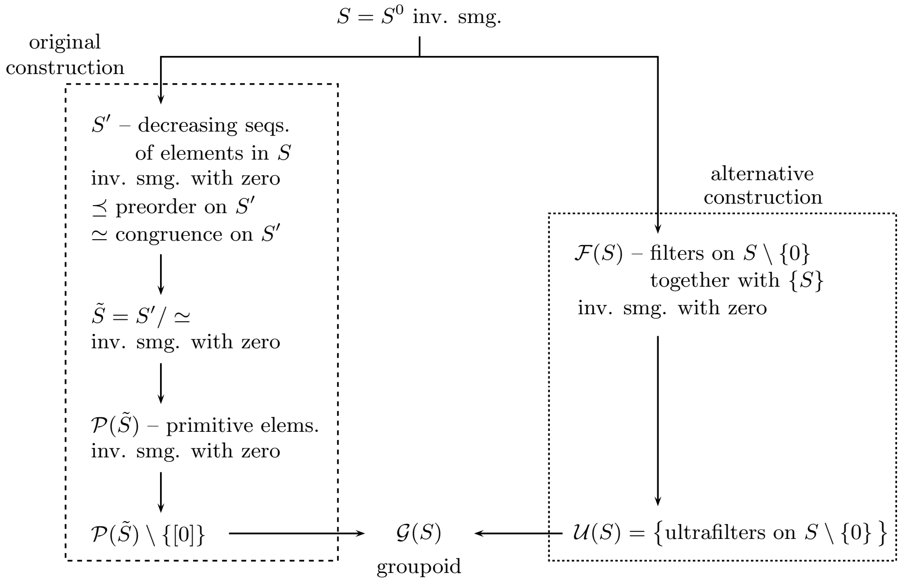
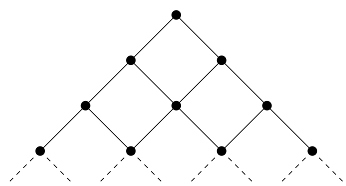


Figure 9.1: Alternative approach to the topological groupoid

Among other applications of the topological groupoid, Kellendonk proves the following statement:

Theorem 1 ([24], Theorem 6). *The finite type tilings $\mathcal{T}$ and $\mathcal{T}'$ are topologically equivalent if and only if their topological groupoids $\mathcal{G}(S(\mathcal{T}))$ and $\mathcal{G}(S(\mathcal{T}'))$ are isomorphic.*

Can the ultrafilter approach be used to give an easier proof of this result? In particular in dimension 1, the proper ultrafilters have a rather nice description, for it can be proved that they consist on the infinite binary trees with maximum element and in which every element has precisely two descendents; that is, the trees of the form



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Bibliography

[1] J. M. Autebert, J. Berstel, L. Boasson, *Context-free languages and pushdown automata*, in "Handbook of Formal Languages", G. Rozenberg, A. Salomaa (Eds.), Springer-Verlag, Berlin (1997), 41–110.

[2] M.-P. Béal, F. Mignosi, A. Restivo, *Minimal forbidden words and symbolic dynamics*, in "Symposium on Theoretical Aspects of Computer Science", C. Puech, R. Reischuk (Eds.), LNCS 1046, Springer (1996), 555–566.

[3] D. Beauquier, M. Nivat, *About rational sets of factors of a bi-infinite word*, in "Automata, Languages and Programming", Lecture Notes in Computer Science, **194**, Springer-Verlag, Berlin (1985), 33–42.

[4] J. Bellissard, *Schrödinger's operators with an almost periodical potential*, in "Mathematical Problems in Theorectical Physics", R. Schrader, R. Seiler (Eds.), Lecture Notes in Physics, **153**, Springer Verlag, Berlin (1982), 356–359.

[5] R. H. Bruck, "A survey of binary systems", Ergebnisse der Math., Neue Folge, Vol. 20, Berlin, Springer (1958).

[6] S. Bulman-Fleming, J. Fountain, V. Gould, *Inverse semigroups with zero: covers and their structure*, J. Austral. Math. Soc. (Series A), **67** (1999), 15–30.

[7] C. Carvalho, "Generation and presentations for semigroup constructions: Bruck-Reilly extensions and P-semigroups", Ph.D. Thesis, University of St. Andrews (2005).

[8] A. H. Clifford, *A class of d-simple semigroups*, American Journal of Mathematics, **75** No. 3 (1953), 547–556.

[9] A. H. Clifford, G. B. Preston, "The algebraic theory of semigroups", Vol. I, Mathematical Surveys No. 7 Part 1, American Mathematical Society, Providence (1967).

[10] A. H. Clifford, G. B. Preston, "The algebraic theory of semigroups", Vol. II, Mathematical Surveys No. 7 Part 2, American Mathematical Society, Providence (1967).

[11] Michael R. Darnel, "Theory of lattice-ordered groups", Monographs and textsbooks in pure and applied mathematics, Marcel Dekker, Inc., New York (1995).

[12] E. Dombi, N. Gilbert, *The tiling semigroups of one-dimensional periodic tilings*, J. Austral. Math. Soc., **87** (2009), 153–160.

[13] E. R. Dombi, N. D. Gilbert, *HNN constructions for free inverse semigroups and Bruck-Reilly extensions*, unpublished preprint.

[14] E. R. Dombi, N. D. Gilbert, *F^*-inverse covers for tiling semigroups*, submitted for publication.

[15] P. Dubreil, *Sur les problèms d'immersion e la théorie des modules*, C. R. Acad. Sci. Paris, **216** (1943), 625–627.

[16] M. C. Escher, B. Ernst, "The Magical Mirror of M. C. Escher", Taschen, Berlin (1978).

[17] D. Giammarresi, A. Restivo, *Two-dimensional languages*, in "Handbook of Formal Languages, Vol. 3", G. Rozenberg , A. Salomaa (Eds.), Springer-Verlag, Berlin (1997), 215–267.

[18] G. Gordon, *The answer is* $2^n\, n!$. *What is the question?*, American Mathematical Monthly, **106** (1999), 636–645.

[19] B. Grunbaum, C. S. Shephard, "Tilings and Patterns — An Introduction", W. H. Freeman and Company, New York (1989).

[20] J. M. Howie, "Automata and Languages", Oxford Science Publications, Oxford, Clarendon Press (1991).

[21] J. M. Howie, "Fundamentals of semigroup theory", London Mathematical Society Monographs No. 12, Oxford, Clarendon Press (1995).

[22] D. L. Johnson, "Presentations of Groups", London Mathematical Society, Student Texts **15**, Cambridge University Press, UK (1997).

[23] J. Kellendonk, *Noncommutative geometry of tilings and gap labelling*, Reviews in Mathematical Physics, **7** No. 7 (1995), 1133–1180.

[24] J. Kellendonk, *The local structure of tilings and their integer group of coinvariants*, Comm. Math. Phys., **187** (1997), 115–157.

[25] J. Kellendonk, *Topological equivalence of tilings*, J. Math. Phys., **38** No. 4 (1997), 1823–1842.

[26] J. Kellendonk, M. V. Lawson, *Tiling semigroups*, Journal of Algebra, **224** (2000), 140–150.

[27] J. Kellendonk, I. F. Putnam, *Tilings, C^*-algebras and K-theory*, in "Directions in Mathematical Quasicrystals", M. Baake, R. Moody (Eds.), CRM Monograph Series, AMS, Providence (RI) (2000), 177–206.

[28] J. Kellendonk, M. V. Lawson, *Universal groups for point-sets and tilings*, Journal of Algebra, **276** (2004), 462–492.

[29] B. P. Kochin, *The structure of inverse ideal-simple ω-semigroups*, Vestnik Leningrad, **23** No. 7 (1968), 41–50.

[30] S. Lang, "Algebra", Third Edition, Addison-Wesley Publishing Company, Reading (1993).

[31] M. V. Lawson, *McAlister semigroups*, Journal of Algebra **202** (1998), 276–294.

[32] M. V. Lawson, "Inverse semigroups: The Theory of Partial Symmetries", World Scientific, Singapore (1998).

[33] M. V. Lawson, *One-dimensional tiling semigroups*, Semigroup Forum, **68** (2004), 159–176.

[34] Daniel H. Lenz, *On an order-based construction of a topological groupoid from an inverse semigroup*, Proceedings of the Edinburgh Mathematical Society, **51** (2008), 387–406.

[35] D. Levine, P. J. Steinhardt, *Quasicrystals: A new class of ordered structures*, Phys. Rev. Lett., **53** No. 15 (1984), 2477–2480.

[36] S. Margolis, J. C. Meakin, *E-unitary inverse monoids and the Cayley graph of a group presentation*, Journal of Pure and Applied Algebra, **58** No. 1 (1989), 45–76.

[37] T. Masuda, J. Morita, *Local properties, bialgebras, and representations for one-dimensional tilings*, J. Phys. A: Math. Gen., **37** (2004), 2661–2669.

[38] D. B. McAlister, *Groups, semilattices and inverse semigroups, II*, Transactions of the American Mathematical Society, **196** (1974), 351–370.

[39] D. McAlister, N. R. Reilly, *E-unitary covers for inverse semigroups*, Pacific Journal of Mathematics, **68** No. 1 (1977), 161–174.

[40] D. B. McAlister, *Finite nilpotent semigroups, Ash's theorem and one-dimensional tilings*, manuscript of talk given at CAUL (October 19, 2001) and at Northern Illinois University (Jully 26, 2002).

[41] D. B. McAlister, *Inverse semigroups and tilings*, manuscript of talk given at CAUL (April 18, 2003).

[42] D. B. McAlister, *An introduction to E^*-unitary inverse semigroups — from an old-fashioned perspective*, in “Proceedings of the Workshop Semigroups and Languages”, I. M. Araújo, M. J. J. Branco, V. H. Fernandes, G. M. S. Gomes (Eds.), Singapore, World Scientific (2004), 133–150.

[43] D. B. McAlister, F. Soares, *One-dimensional tiling semigroups and factorial languages*, Communications in Algebra, **37** No. 1 (2009), 276–295.

[44] D. B. McAlister, F. Soares, *Tiling semigroups of n-dimensional hypercubic tilings*, accepted for publication in Communications in Algebra.

[45] W. D. Munn, *Regular ω-semigroups*, Glasgow Math. J., **9** (1968), 46–66.

[46] W. D. Munn, *On simple inverse semigroups*, Semigroup Forum, **1** (1970), 63–74.

[47] W. D. Munn, *A note on E-unitary inverse semigroups*, Bull. London Math. Soc., **8** (1976), 71–76.

[48] A. L. T. Paterson, “Inverse semigroups, groupoids, and their operator algebras”, Pure and applied mathematics, John Wiley & Sons, New York (1998).

[49] R. Penrose, *The role of aesthetics in pure and applied mathematical research*, Bull. Inst. Math. Appl., **10** Nos. 7&8 (1974), 266–271.

[50] D. Perrin, *Words*, “Combinatorics on words”, M. Lothaire, Cambridge University Press (1997), 1–17.

[51] M. Petrich, “Inverse semigroup”, Mathematical Surveys No. 7 Part 2, American Mathematical Society, Providence (1967).

[52] N. R. Reilly, *Bisimple ω-semigroups*, Pro. Glasgow Math. Assoc., **7** No. 1 (1966), 160–167.

[53] N. R. Reilly, A. H. Clifford, *Bisimple inverse semigroups as semigroups of ordered triples*, Canadian Journal of Mathematics, **20** No. 1 (1968), 25–39.

[54] N. Reilly, *Bisimple inverse semigroups*, Transactions of the American Mathematical Society, **132** No. 1 (1968), 101–114.

[55] N. Ruškuc, “Semigroup presentations”, Ph.D. Thesis, University of St. Andrews (2000).

[56] B. Schein, *Semigroups of strong subsets*, Volzhskii Matematiskii Sbornik, **4** (1966), 180–186.

[57] B. Schein, *Free inverse semigroups are not finitely presented*, Acta Math. Acad. Sci. Hung, **26** (1975), 41–52.

[58] F. Soares, “Semigrupos de Pavimentações”, Masters Thesis, University of Lisbon (2005).

[59] B. Steinberg, *The uniform word problem for groups and finite Rees quotients of E-unitary inverse semigroups*, Journal of Algebra, **266** No. 1 (2003), 1–13.

[60] A. Yamamura, *Presentations of Bruck-Reilly extensions and decision problems*, Semigroup Forum, **62** (2001), 79–97.

[61] Y. Zhu, *Some fundamental properties of tiling semigroups*, Journal of Algebra, **252** (2002), 195–204.

Index